

GEOMETRY AND ARITHMETIC ON A QUINTIC THREEFOLD

CATERINA CONSANI[†] AND JASPER SCHOLTEN

INTRODUCTION

In this paper we investigate the geometry and the arithmetic of a non-rigid Calabi-Yau threefold \overline{X} with 120 nodes. This variety is the projective closure in $\mathbf{P}_{\mathbb{C}}^4$ of a special fiber in a family of affine threefolds. The family is obtained as a self-fiber product of a fibration of affine curves defined by generalized Chebyshev polynomials in two variables. These polynomials are particular kind of symmetric functions with the remarkable property of taking only three different values at their critical points. They naturally generalize in higher dimensions the classical Chebyshev polynomials in one variable. The self-fiber product of a one parameter family of degree five generalized Chebyshev polynomials $P_5(x_1, x_2) = t$ is an affine fibration endowed with an action of the dihedral group of order eight. The fiber at the origin X has only non-degenerate singular points as singularities. The presence of a group action on the fibers of the self-fiber product (and in particular on X) makes possible the computation of the Hodge numbers of a desingularization \tilde{X} of \overline{X} . Computations show that \tilde{X} is a Calabi-Yau threefold with $h^2(\tilde{X}) = h^{1,1}(\tilde{X}) = 141$ and $h^3(\tilde{X}) = 4$ (*cf.* corollaries 3.7 and 3.10).

The particular position of the singular points on X —that is a consequence of the symmetries of the Chebyshev polynomials—allows the presence of a divisor on the threefold through the nodes and not homologous to a multiple of a generic hyperplane section. This cycle is the responsible for the non-vanishing of $h^{2,1}(\tilde{X})$.

The affine threefold X has a covering Y whose definition is given in terms of Dickson polynomials. More precisely, X is the variety of the orbits for the action of (two copies of) the symmetric group in three variables on Y . Although the geometry of Y is much more complicated than that of X (*e.g.* its singular locus consists of 8750 points and unlikely X , Y acquires further singularities at infinity), the presence of this covering plays a relevant role in the proof of the existence of correspondences on X . These correspondences provide a reasonable explanation for some of the arithmetic results we prove in the paper.

Over the field of the rational numbers, \tilde{X} has good reduction outside the set $\{2, 3, 5\}$ (*cf.* section 4). One of the main goals of this paper is the investigation of the Galois representation $\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}(H^3(\tilde{X}_{\overline{\mathbb{Q}}}, \mathbb{Q}_{\ell}))$ and the L-function $L(H^3(\tilde{X}_{\overline{\mathbb{Q}}}, \mathbb{Q}_{\ell}), s)$ (ℓ a rational prime) of the resolved variety \tilde{X} . This study is rather complicated and it is accomplished in few steps. First, we compute the local Euler factors at many (rational) primes of good reduction (*cf.* paragraph 5). This is done by counting the points of \tilde{X} over various finite fields and using the Lefschetz trace formula. The shape of these

[†]Partially supported by the NSF grant DMS-9701302 and by the NSERC grant 72016789.

polynomials and their splitting behavior in the real quadratic extension $F = \mathbb{Q}(\sqrt{5})$, clearly suggest that the scalars extension $\rho \otimes \mathbb{Q}_\ell(\sqrt{5})$ is induced by a two-dimensional representation σ of the Galois group $\text{Gal}(\overline{\mathbb{Q}}/F)$. We prove this in paragraph 6 (*cf.* theorem 6.2). The existence of algebraic correspondences on X defined over F (*cf.* the discussion at the end of paragraph 6), provides a reason for the reducibility of the restriction $(\rho \otimes \overline{\mathbb{Q}_\ell})|_{\text{Gal}(\overline{\mathbb{Q}}/F)}$ and another explanation for the induceness.

Following the theory of Jacquet and Langlands one would predict the modularity of σ , namely the existence of a Hilbert modular form \mathfrak{f} over F whose associated Galois representation $\sigma_{\mathfrak{f}}$ is equivalent to σ . We deduce the expected weight $\mathbf{k} = (2, 4)$ and the level $\mathfrak{N} = 5$ of \mathfrak{f} from the shape of the (conjectural) functional equation of the L-function of the threefold (*cf.* paragraph 7). This procedure is possible because one knows the invariance of the L-function for induced representations. With the help of a computer, we tested numerically the functional equation, under the assumption of its analytic continuation.

The existence of a Hilbert cusp form with the correct behavior is deduced from the theorem of Jacquet and Langlands. The properties of \mathfrak{f} are characterized on the automorphic side, by those of the corresponding automorphic representation $\pi_{\mathfrak{f}}$ of $G_B(\mathbf{A}_F)$: the adeles of the algebraic group G_B associated to (the multiplicative group of) a definite quaternion algebra B over F . The definition of this algebra and the description of a correlated Eichler order are derived from the expected description of the conductor in the functional equation (*cf.* section 8).

The action of the Hecke operators on the space of the quaternionic forms is given by the Brandt matrices. These square matrices are naturally associated to an Eichler order and they describe the representation of the Hecke operators acting on a space of theta series attached to the norm form of the algebra B . In our case, it is known (*cf.* [30]) that these series provide a basis for the space of Hilbert modular forms to which \mathfrak{f} belongs. In the paper we provide an explicit description of the algorithm used for the definition of the Brandt matrices (*cf.* paragraph 11). It is known that these matrices generate a commutative semisimple ring and they satisfy the same identities fulfilled by the classical Hecke operators. Hence, through a process of simultaneous diagonalization of them we provide a proof of the existence of a Hilbert modular form \mathfrak{f}' with properties analogous to those of \mathfrak{f} (*cf.* paragraph 11). The function \mathfrak{f}' is represented by a common eigenvector of the Brandt matrices and its eigenvalues are known to be equal to the traces of Frobenius of the associated Galois representation $\sigma_{\mathfrak{f}'}$.

In our case, all the computed eigenvalues are in fact equal to the traces of Frobenius of the geometric Galois representation σ . This suggests that σ and $\sigma_{\mathfrak{f}'}$ are isomorphic upto semisimplification. By a theorem of Faltings (*cf.* [9]), two semisimple Galois representations are isomorphic if their traces of Frobenius are equal for sufficiently many primes within a finite set. Unfortunately, we are not able to make this test set small enough to be contained in the set of primes for which we can compute the traces. A method described in [18] allows one to make the test set very small if one knows that the traces of Frobenius are even. In our case all traces appear to be even, but we can only prove that for the geometric Galois representation σ .

There are at least two further problems that we have not investigated in this paper but whose study seems natural to pursue.

The very rich geometry of \tilde{X} makes this variety a good example for testing some of the open conjectures in arithmetic and in the study of the algebraic cycles. The theory of Jacquet and Langlands predicts the modularity of the 4-dimensional selfdual Galois representation ρ . More precisely, we expect to support evidence for the existence of a Siegel modular form of weight 3 and genus 2, defined on some subgroup $\Gamma_0(N)$ of $\mathrm{Sp}(4, \mathbb{Z})$, whose associated Galois representation is equivalent to ρ . It is conceivable, although we have not checked this in details, that this modular form arises as (theta) lifting of the Hilbert modular form f' .

A second theme of investigation is the verification of Bloch's conjecture that relates in our case, the order of zero of the L-function at the center of the critical strip and the nonvanishing of the Griffiths group of the algebraic cycles on \tilde{X} homologous to zero modulo algebraic equivalence. From the numerical analysis made on the functional equation, we were able to test the vanishing of the L-function $L(H^3, s)$ at $s = 2$, exactly on the first order. In accord with Bloch's conjecture, this first order of zero would be explained geometrically by the presence of a codimension-two algebraic cycle on \tilde{X} , whose class in the Griffiths group is not trivial. We expect that the construction described by Kimura in [14] may be helpful for the definition of this cycle.

In the following, we give a brief account on the arguments illustrated in the various paragraphs.

In the first two sections we define the threefold fibration in which X appears as a special fiber and we compare our construction with others made in [28] and [29]. In the third paragraph we describe the group action on X using which we compute the Hodge numbers. The sections 4,5,6 and 7 include the study of the L-function and the Galois representation of \tilde{X} . The paragraphs 8,9 are dedicated to the definition and the properties of the quaternion algebra B and the Eichler order. The final sections 10,11 report on the description of the algorithm used for the definition of the Brandt matrices and for their simultaneous diagonalization.

In the paper we have included few tables which resume the high number of computations we have made. In many situations, the help of an appropriate computer program (Mathematica, Maple, Pari, Magma) was fundamental.

Acknowledgments. This paper is the result of a rather long period of research dedicated to the study of the Langlands program. Our intention was to approach this complicated theory via the study of a hypersurface whose geometry and arithmetic turned out to be very rich. In many occasions, we referred to the experienced advice of different mathematicians to obtain answers or for explaining our computations and listen their comments. This is a heartfelt thank to all who have been contacted by us either in person or by e-mail. Among these people we would like to mention Juliusz Brzezinski and Elise Bjerkholdt to whom we often referred for questions regarding the theory of orders in quaternion algebras. We thank Fred Diamond who explained to us the approach to the study of the Hilbert modular forms via functions defined on a quaternion algebra. Many thanks are due to Dinakar Ramakrishnan for his advice and because he

suggested the lecture of the thesis [24]. This paper became for us an inestimable source of reference, especially in the process of setting up the algorithm for the computation of the Brandt matrices. Our thanks go to Spencer Bloch and Jaap Top for the support and the encouragement we received from them in these years and to Jim Cogdell, Pierre Deligne and Bert van Geemen who, in many occasions, patiently listened our ideas and replied with interesting remarks. The first author gratefully thanks the Max-Planck Institut für Mathematik and the Institut des Hautes Études Scientifiques for the kind hospitality received during her stay there, respectively in June 1999 and January 2000. She also acknowledges the support received by the Institute for Advanced Study in the year 1999-2000. The second author wishes to thank the first author for asking him to join this project, and for the hospitality received during various visits.

1. A SKEW-PENTAGON CONFIGURATION OF LINES IN THE AFFINE PLANE.

In this paragraph we review the description of a family of affine, plane, quintic curves introduced by van Geemen and Werner (*cf.* [28] and [29]) in relation with the study of a particular configuration of lines in the plane. The authors called this configuration a skew pentagon (five lines meeting only in pairs and stable under the involution $(x, y) \mapsto (x, -y)$) as it generalizes the construction of the regular pentagon defined by Hirzebruch in [11]. The equation of a skew pentagon depends upon two parameters a, b as follows

$$F_{a,b}(x, y) = (x + a)(y^2 - x^2)(y^2 - b(x + 1)^2).$$

It is easy to verify that $F_{a,b}$ has 10 critical points (*i.e.* all of its partial derivatives vanish) where the five lines mutually intersect. The polynomial $F_{a,b}$ has 2 further critical points on the x -axis. If one adds the condition that

(1.1)

the value of $F_{a,b}$ at the remaining 4 critical points (not on the lines) is the same,

it turns out that the parameters a, b must satisfy the quartic equation

$$(1.2) \quad a^2(b - 1)^2 - 2ab(b - 1) + b(b - 5) = 0.$$

Van Geemen and Werner noticed that the solutions of (1.2) are parametrized via the introduction of an additional parameter t

$$a = \frac{t(t + 5)}{t^2 - 5}, \quad b = \frac{t^2}{5}.$$

Set $H_t(x, y) := F_{a(t), b(t)}(x, y)$. Then, $H_t(x, y)$ defines a family of plane curves parametrized by the rational quartic (1.2). Each element of this family defines a skew pentagon configuration of lines in the affine plane under the condition (1.1).

It is easy to check that for a generic value of t (*i.e.* for all t except finitely many of them), the quintic threefold $V \subset \mathbf{P}^4$ defined as the closure of the affine variety in \mathbf{A}^4

$$(1.3) \quad H_t(x, y) - H_t(u, v) = 0$$

has 118 nodes (rational double points): *i.e.* $10 \cdot 10 + 4 \cdot 4 + 1 \cdot 1 + 1 \cdot 1$. These points are the only kind of singularities of V and they are all located on an affine chart (*i.e.* the affine threefold defined by the equation (1.3) does not acquire further singular

points in its closure). The condition (1.1) constrains the critical values taken by the function $H_t(x, y)$ (for a generic choice of t) to be only four. In [28] it is shown that a desingularization \tilde{V}_t of the generic fiber of the family (1.3), obtained by blowing up along its singular locus has $h^3 = \dim H^3(\tilde{V}_t, \mathbb{Q}) = 6$ ($h^{3,0} = 1 = h^{0,3}$) and $h^2 = \dim H^2(\tilde{V}_t, \mathbb{Q}) = h^{1,1} = 138$.

Within the family (1.3) there are two special fibers with a different number of singular points and different Hodge numbers. The first, is the fiber at

$$t = -5 \pm 2\sqrt{5}, \quad \text{i.e. } a = \frac{1}{2}, \quad b = 9 \pm 4\sqrt{5}.$$

This threefold has 126 nodes and it is isomorphic to the one studied by Hirzebruch (*i.e.* the self product of a regular pentagon configuration). Its desingularization has $h^3 = 2$ ($h^{2,1} = 0 = h^{1,2}$) and $h^2 = h^{1,1} = 152$ (*cf.* [11] and [29]). It was shown in [28] that this variety is equivalent to the fiber at $c = \frac{1}{2}$ in the family of threefolds obtained as self-product of the family

$$(1.4) \quad F_c(x, y) = (x + c)(y^4 - y^2(2x^2 - 2x + 1) + \frac{1}{5}(x^2 + x - 1)^2).$$

The generic fiber of (1.4) is a skew pentagon but this family satisfies the condition (1.1) only for $c \in \{\frac{1}{2}, -2\}$. The second special fiber in (1.3) is obtained for

$$t = 5 \pm 2\sqrt{5}, \quad \text{i.e. } a = \frac{1 \pm \sqrt{5}}{2}, \quad b = 9 \pm 4\sqrt{5}$$

(each couple must be chosen with the same sign). This threefold is isomorphic to the fiber at $c = -2$ in (1.4). These varieties have 120 nodes: *i.e.* $10 \cdot 10 + 4 \cdot 4 + 2 \cdot 2$. The two further singular points (with respect to the singular set of the generic fiber) come up because $H_t(x, t)$ takes the same value also at the 2 critical points on the x -axis not on the lines (note that 2 among the 10 critical points of $F_{-2}(x, y)$ defined by the intersection points of the lines are on the x -axis *cf.* (2.7) below). The desingularized varieties have $h^3 = 4$ ($h^{3,0} = h^{0,3} = h^{2,1} = h^{1,2} = 1$) and $h^2 = h^{1,1} = 141$. This paper will focus on the geometry and arithmetic of them.

2. THE CHEBYSHEV FAMILY OF QUINTIC THREEFOLD.

In this section we introduce a family of quintic threefolds in \mathbf{P}^4 whose geometry (and arithmetic) is related to the special fiber at $t = 5 \pm 2\sqrt{5}$ in (1.3).

We start by introducing the following function in three variables

$$(2.1) \quad f(y_1, y_2, y_3) := y_1^5 + y_2^5 + y_3^5 + (y_1 y_2)^5 + (y_1 y_3)^5 + (y_2 y_3)^5.$$

This polynomial is the sum of two Dickson polynomials of the first kind: $D_5^{(i)}(x_1, x_2, 1)$, $i = 1, 2$. We recall, for completeness, the definition of Dickson polynomials of the first kind in several variables and few of their properties. For a complete report on this theory we refer to [17].

Let R be a commutative ring with identity.

Definition 2.1. *The Dickson polynomials of the first kind $D_n^{(i)}(x_1, x_2, \dots, x_k, a)$, $1 \leq i \leq k$, $a \in R$ $k \geq 1$ integer, are given by the functional equation*

$$D_n^{(i)}(x_1, x_2, \dots, x_k, a) = s_i(y_1^n, \dots, y_{k+1}^n), \quad 1 \leq i \leq k,$$

where $x_i = s_i(y_1, \dots, y_{k+1})$ and $y_1 \cdots y_{k+1} = a$.

The vector $D(k, n, a) = (D_n^{(1)}, \dots, D_n^{(k)})$ of the k Dickson polynomials is called the Dickson polynomial vector.

When $k = 2$, the Dickson polynomials can be defined recursively in the following way

Lemma 2.1. *The polynomials $D_n^{(i)} := D_n^{(i)}(x_1, x_2, a)$, $i = 1, 2$ satisfy the recurrence relation*

$$D_n^{(1)} = x_1 D_{n-1}^{(1)} - x_2 D_{n-2}^{(1)} + a D_{n-3}^{(1)}$$

with $D_0^{(1)} = 3$, $D_1^{(1)} = x_1$, $D_2^{(1)} = x_1^2 - 2x_2$

and

$$D_n^{(2)} = x_2 D_{n-1}^{(2)} - a x_1 D_{n-2}^{(2)} + a^2 D_{n-3}^{(2)}$$

with $D_0^{(2)} = 3$, $D_1^{(2)} = x_2$, $D_2^{(2)} = x_2^2 - 2a x_1$.

Proof. cf. [17]. □

When $k = 2$ and $a = 1$, one has

$$(2.2) \quad D_5^{(i)}(x_1, x_2, 1) = s_i(y_1^5, y_2^5, y_3^5), \quad 1 \leq i \leq 2$$

$$x_i = s_i(y_1, y_2, y_3), \quad \text{and} \quad y_1 y_2 y_3 = 1$$

$$s_1 = y_1 + y_2 + y_3, \quad s_2 = y_1 y_2 + y_1 y_3 + y_2 y_3, \quad s_3 = y_1 y_2 y_3.$$

The sum $D_5^{(1)}(x_1, x_2, 1) + D_5^{(2)}(x_1, x_2, 1)$ is then the polynomial f in (2.1). The recurring relation shows that f may be described by a symmetric equation of degree 5 in the variables x_1, x_2 (x_i are the first two elementary symmetric polynomials in three variables $x_i = s_i(y_1, y_2, y_3)$, under the condition $y_1 y_2 y_3 = 1$). It turns out that f is implicitly defined by

$$f(y_1, y_2, y_3) = D_5^{(1)}(x_1(y_1, y_2, y_3), x_2(y_1, y_2, y_3), 1) + D_5^{(2)}(x_1(y_1, \dots), x_2(y_1, \dots), 1)$$

$$D_5^{(1)}(x_1, x_2, 1) + D_5^{(2)}(x_1, x_2, 1) = P_5(x_1, x_2),$$

where $P_5(x_1, x_2)$ is the following symmetric quintic polynomial

$$(2.3) \quad P_5(x_1, x_2) = (x_1^5 + x_2^5) - 5x_1 x_2 (x_1^2 + x_2^2) + 5x_1 x_2 (x_1 + x_2) + 5(x_1^2 + x_2^2) - 5(x_1 + x_2).$$

Note that (2.3) can be written as a sum of two determinants (each determinant corresponds to a Dickson polynomial)

$$P_5(x_1, x_2) = \det \begin{pmatrix} x_1 & 1 & 0 & 0 & 0 \\ 2x_2 & x_1 & 1 & 0 & 0 \\ 3 & x_2 & x_1 & 1 & 0 \\ 0 & 1 & x_2 & x_1 & 1 \\ 0 & 0 & 1 & x_2 & x_1 \end{pmatrix} + \det \begin{pmatrix} x_2 & 1 & 0 & 0 & 0 \\ 2x_1 & x_2 & 1 & 0 & 0 \\ 3 & x_1 & x_2 & 1 & 0 \\ 0 & 1 & x_1 & x_2 & 1 \\ 0 & 0 & 1 & x_1 & x_2 \end{pmatrix}.$$

Each determinant represents a generalized Chebyshev polynomial (of the first kind) associated to the root system A_2 (cf. [13]). The general definition of a Chebyshev polynomial of the first kind in k indeterminates is given by (cf. [14])

$$P_{m,n}^{(-1/2)}(x_1, \dots, x_k) = \sum_{i=1}^{k+1} \sum_{\substack{j=1 \\ j \neq i}} u_i^m u_j^{-n}, \quad \text{for } m, n \in \mathbb{Z}$$

$$x_i = s_i(y_1, \dots, y_{k+1}), \quad \text{and } x_{k+1} = y_1 \cdots y_{k+1} = 1.$$

Hence, one may write f as the sum $P_{0,-5}^{(-1/2)}(x_1, x_2) + P_{0,5}^{(-1/2)}(x_1, x_2)$, where the variables x_1, x_2, x_3 are defined as in (2.2).

Dickson's polynomials share a number of interesting properties, especially when the ring R is the finite field \mathbb{F}_q with q elements. We recall one of these for future reference.

The Dickson polynomial vector $D(k, n, a)$ induces a map of \mathbb{F}_q^k into itself defined by

$$D(k, n, a) : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^k$$

$$(\zeta_1, \dots, \zeta_k) \mapsto (D_n^{(1)}(\zeta_1, \dots, \zeta_k, a), \dots, D_n^{(k)}(\zeta_1, \dots, \zeta_k, a)).$$

The following result shows under what condition $D(k, n, 0)$ permutes \mathbb{F}_q^k .

Proposition 2.2. *$D(k, n, 0)$ permutes \mathbb{F}_q^k if and only if*

$$\gcd(n, q^s - 1) = 1, \quad \text{for } s = 1, \dots, k.$$

Proof. We refer to [17] Theorem 3.41 for a complete proof. In *op.cit.*, it is shown under which condition the Dickson vector $D(k, n, a)$ induces permutation on the field \mathbb{F}_q , for $a \in \mathbb{F}_q^\times$. \square

In the rest of this paragraph we will consider the polynomial $P_5(x_1, x_2)$ as a polynomial of degree 5 in the variables x_1, x_2 , “forgetting” the description of these variables in terms of elementary symmetric polynomials. The interpretation of P_5 in terms of Dickson's polynomials will be reconsidered later on in the paper.

The function $P_5(x_1, x_2)$ has 16 non degenerate critical points and its highest homogeneous part is non-degenerate (*i.e.* it has only one critical point at zero). A very important property of this polynomial is that of taking only three different values at its critical points. One can easily verify that P_5 has 10 critical points with critical value -2 (these occur at the intersection points of the lines), 2 critical points with critical value 6 (they occur on the x_1 -axis) and 4 critical points with value -3. This particular behavior characterizes the Chebishev polynomials $P_d(x_1, x_2)$ of degree d , independently of the choice of d . In [7], Chmutov defines the number of critical points associated to

a given critical value, for any positive d . In this paper we are mainly interested in the case $d = 5$, so we will skip the general description for self-contained reasons.

Starting with the Chebyshev polynomial $P_5(x_1, x_2) \in \mathbb{Q}[x_1, x_2]$, let first define the fibration of affine plane curves (generically of genus 6)

$$(2.4) \quad \mathbf{A}^2 \rightarrow \mathbf{A}^1, \quad P_5(x_1, x_2) = t.$$

Then, we introduce a new couple of variables $\{x_4, x_5\}$ and we consider the self-fiber-product of the family (2.4). We obtain the following family of quintic hypersurfaces in $\mathbf{A}_{\mathbb{Q}}^4$

$$(2.5) \quad \mathbf{A}^4 \rightarrow \mathbf{A}^1, \quad f_5(x_1, x_2, x_4, x_5) = t$$

$$f_5(x_1, x_2, x_4, x_5) := P_5(x_1, x_2) - P_5(x_4, x_5).$$

Let $\bar{X} \subset \mathbf{P}^4$ be the projective closure of the fiber at $t = 0$ in (2.5). Then, it follows from the definition of (and the value at) the critical points that

Lemma 2.3. *The projective treefold \bar{X} has*

$$(10)^2 + (4)^2 + (2)^2 = 120$$

non-degenerate double points.

Note that the fibers at $t = 5 \pm 2\sqrt{5}$ in (1.3) and at $c = -2$ in (1.4) are threefolds with the same number of double points. This analogy is not casual. To explain it, we describe more in detail the Chebyshev family.

In the cyclotomic field $\mathbb{Q}(\zeta_5)$ (ζ_5 a 5-th primitive root of unit), the contraction-translation of $P_5(x_1, x_2)$: $-\frac{1}{10}P_5(x_1, x_2) + 2$ factorizes as

$$(2.6) \quad -\frac{1}{10}P_5(x_1, x_2) + 2 = -\frac{1}{10}(x_2 + \zeta_5 + \zeta_5^3 - x_1 - x_1\zeta_5 - x_1\zeta_5^2 - x_1\zeta_5^3)(x_2 - 1 - \zeta_5 - \zeta_5^3 + x_1\zeta_5)$$

$$(x_2 - 1 - \zeta_5 - \zeta_5^2 + x_1\zeta_5^2)(x_2 + \zeta_5 + \zeta_5^2 + x_1\zeta_5^3)(x_2 + 2 + x_1).$$

In other words, $\mathbb{Q}(\zeta_5)$ is the minimal field extension of \mathbb{Q} over which every fiber of (2.4) splits into a product of 5 lines. Furthermore, the maximal real subfield of the cyclotomic field $\mathbb{Q}(\zeta_{20})$ is the minimal field extension over which each fiber of the skew-pentagon configuration $F_{-2}(x_1, x_2)$ is defined. Let $\zeta_{re} = \zeta_{20} + \zeta_{20}^{-1}$. Then, in the extension $\mathbb{Q}(\zeta_{re})$ the five lines are given by

$$(2.7) \quad F_{-2}(x_1, x_2) = \frac{1}{625}(5x_2 + x_1\zeta_{re}^3 + 5\zeta_{re} - 2\zeta_{re}^3)(5x_2 + 5\zeta_{re} - \zeta_{re}^3 - 10x_1\zeta_{re} + 3x_1\zeta_{re}^3)$$

$$(5x_2 - 5\zeta_{re} + \zeta_{re}^3 + 10x_1\zeta_{re} - 3x_1\zeta_{re}^3)(-x_1\zeta_{re}^3 - 5\zeta_{re} + 2\zeta_{re}^3 + 5x_2)(x_1 - 2)$$

The following proposition shows the existence of an isomorphism over $\mathbb{Q}(i)$ which identifies the family (2.4) and the skew-pentagon family (1.4): $F_{-2}(x_1, x_2) = t$, (for $c = -2$).

Proposition 2.4. *Let $F_{-2}(x_1, x_2) = (x_1 - 2)(x_2^4 - x_2^2(2x_1^2 - 2x_1 + 1) + \frac{1}{5}(x_1^2 + x_1 - 1)^2)$ and $P_5(x_1, x_2) = x_1^5 + x_2^5 - 5(x_1x_2 - 1)(x_1^2 + x_2^2 - x_1 - x_2)$. Then*

$$P_5(x_1, x_2) = -10F_{-2}(x_1, x_2) - 2$$

by means of the $\mathbb{Q}(i)$ -linear map

$$\mathbf{A}_{\mathbb{Q}}^2 \rightarrow \mathbf{A}_{\mathbb{Q}}^2, \quad (x_1, x_2) \mapsto \left(-\frac{x_1 + x_2}{2} + 1, \frac{i(x_1 - x_2)}{2}\right).$$

Proof. It is rather easy to check that the affine 3-folds $F_{-2}(x_1, x_2) - F_{-2}(x_3, x_4) = 0$ and $P_5(x_1, x_2) - P_5(x_3, x_4) = 0$ have the same number of points over fields containing $\sqrt{-1}$. Hence, any bijective linear transformation between $P_5(x_1, x_2)$ and $F_{-2}(x_1, x_2)$ will be defined over such fields. Among the critical points of $F_{-2}(x_1, x_2)$, one finds the points $(x_1, x_2) = (\frac{2}{3}a^2 + \frac{1}{2}, a)$ with a satisfying the equation $16a^4 - 36a^2 + 9 = 0$. The field generated by these points is $\mathbb{Q}(\sqrt{5}, \sqrt{3})$. Similarly, among the critical points of $P_5(x_1, x_2)$ one finds the points $(x_1, x_2) = (-b + \frac{1}{2} - \frac{1}{2}b^3, b)$ with b satisfying the equation $b^4 + b^3 + 2b^2 - b + 1 = 0$. The field generated by these points is $\mathbb{Q}(\sqrt{5}, \sqrt{-3})$. Of course, both the polynomials have further critical points, but we simply look for a linear map that relates the above critical points. This is because the image of points like $(-b + \frac{1}{2} - \frac{1}{2}b^3, b)$ (*cf.* above) is defined over a subfield of $\mathbb{Q}(\sqrt{5}, \sqrt{-3}, i) = \mathbb{Q}(b, i)$ and this field contains $\mathbb{Q}(a)$. A similar argument shows that the image of points like $(\frac{2}{3}a^2 + \frac{1}{2}, a)$ is defined over a subfield of $\mathbb{Q}(a, i)$ and this field contains $\mathbb{Q}(b)$. One can easily express the parameter a in terms of b and i : $a = -\frac{i}{4}b^3 - ib + \frac{i}{4}$. The definition of the linear map follows. \square

Remark 2.5.

Over the field $\mathbb{Q}(\zeta_{20})$ (the common field over which each line in both constructions is defined), one may use a cross-ratio method to show that there are only two possible maps between the 2 sets of lines that define *resp.* the Chebyshev and the skew-pentagon families. First, note that 5-tuples of distinct points in general position (*i.e.* no three of them lying on a line) in $\mathbf{P}_{\mathbb{R}}^2$, modulo $\text{PGL}_3(\mathbb{R})$, are the same as 5-tuples of distinct points in $\mathbf{P}_{\mathbb{R}}^1$, modulo $\text{PGL}_2(\mathbb{R})$. In \mathbf{P}^1 one has 2 cross-ratios. Consider the set $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$ as a “basis” in \mathbf{P}^1 and then add the fifth point (a, b) . In this way, a and b will play the role of the two invariants. If the five points in \mathbf{P}^2 are given as the vectors in \mathbf{A}^3 : $v_1 = (v_{11}, v_{12}, v_{13}), \dots, v_5 = (v_{51}, v_{52}, v_{53})$, then we write the 2 ratio invariants as

$$r(v_1|v_2, \dots, v_5) := \frac{D(v_1, v_2, v_4)D(v_1, v_3, v_5)}{D(v_1, v_2, v_5)D(v_1, v_3, v_4)},$$

$$r(v_2|v_3, v_4, v_5, v_1) := \frac{D(v_2, v_3, v_5)D(v_2, v_4, v_1)}{D(v_2, v_3, v_1)D(v_2, v_4, v_5)},$$

where

$$D(v_i, v_j, v_k) := \det \begin{pmatrix} v_{i1} & v_{i2} & v_{i3} \\ v_{j1} & v_{j2} & v_{j3} \\ v_{k1} & v_{k2} & v_{k3} \end{pmatrix}.$$

The two 5-tuples of lines associated to the equations $P_5(x, y) = 0 = F_{-2}(x, y)$, determine 5-tuples of length-3-vectors: v_1, \dots, v_5 and w_1, \dots, w_5 . We computed explicitly the ratios $[r(v_1|v_2, \dots, v_5), r(v_2|v_3, v_4, v_5, v_1)]$ and we compared them with $[r(w_{\sigma(1)}|w_{\sigma(2)}, \dots, w_{\sigma(5)}), r(w_{\sigma(2)}|w_{\sigma(3)}, w_{\sigma(4)}, w_{\sigma(5)}, w_{\sigma(1)})]$ for any permutation of five elements $\sigma : \{1, 2, 3, 4, 5\} \rightarrow \{1, 2, 3, 4, 5\}$. It turns out that there are only two possible matchings, and they correspond to the two isomorphisms over $\mathbb{Q}(i)$ of proposition 2.4. \square

3. GROUP ACTION AND COHOMOLOGY.

In this paragraph we describe a group action on the Chebyshev threefold X (and more in general on any fiber of (2.5)), using which we deduce its Hodge numbers and the topological Euler characteristic.

For resolutions of hypersurfaces singularities in $\mathbf{P}_{\mathbb{C}}^4$ the computation of h^3 depends upon the number of double points s and the defect def , by means of the well known formula: $h^3 = 204 - 2s + 2 \cdot def$. The defect computes the number of independent divisors on the threefold through the nodes and not homologous to a multiple of a generic hyperplane section. Its knowledge depends on the special position of the nodes on the embedded threefold. The defect is a corank of a certain matrix whose size depends on the degree of the hypersurface and on the number of its singular points.

We denote by S the closed set (for the Zariski topology) described by the singular points of X . We write \mathfrak{I}_S for its ideal sheaf. Let $F_5(x_0, \dots, x_4)$ be the homogeneous polynomial whose zeroes define \bar{X} as a hypersurface in \mathbf{P}^4 (we homogenize $f_5(x_1, x_2, x_4, x_5)$ by using a further variable x_0 and then we re-number the variables consecutively). One has the following exact sequence of \mathbb{C} -vector spaces

$$(3.1) \quad 0 \rightarrow H^0(\mathbf{P}^4, \mathfrak{I}_S \mathcal{O}(5)) \rightarrow H^0(\mathbf{P}^4, \mathcal{O}(5)) \xrightarrow{e} H^0(S, \mathcal{O}_S) \rightarrow H^1(\mathbf{P}^4, \mathfrak{I}_S \mathcal{O}(5)) \rightarrow 0$$

where

$$\dim_{\mathbb{C}} H^0(\mathbf{P}^4, \mathcal{O}(5)) = \binom{9}{5} = \frac{9!}{5! \cdot 4!} = 126.$$

The map e evaluates a homogeneous polynomial f_i of degree five at each point P_j of the singular locus S . It can be described by the rectangular matrix $M := (f_i(P_j))$ (whose size is 126×120). The defect of \bar{X} is the corank of M *i.e.* the dimension of $H^1(\mathbf{P}^4, \mathfrak{I}_S \mathcal{O}(5))$.

Because $\dim_{\mathbb{C}} H^0(\mathbf{P}^4, \mathfrak{I}_S \mathcal{O}(5)) \geq 25$ (any linear combination as $\sum_{i=0}^4 L(x_0, \dots, x_4) \cdot \frac{\partial F_5}{\partial x_i}(x_0, \dots, x_4)$ vanishes at S , where L a linear homogeneous polynomial in five variables), one necessarily expects that

$$\text{corank } M = \dim_{\mathbb{C}} H^1(\mathbf{P}^4, \mathfrak{I}_S \mathcal{O}(5)) \geq 19.$$

The evaluation of the rank of $H^0(\mathbf{P}^4, \mathfrak{I}_S \mathcal{O}(5))$, depends upon a precise understanding of the special position of the nodes of X in $\mathbf{A}_{\mathbb{C}}^4$. Note that $\bar{X} - X$ is a non singular variety.

There are three involutions σ_i ($i = 1, \dots, 3$) acting on the affine threefold X and more in general on each fiber of the family (2.5). They are

$$(3.2) \quad \begin{aligned} \sigma_1(x_1, x_2, x_3, x_4) &= (x_2, x_1, x_3, x_4), & \sigma_2(x_1, x_2, x_3, x_4) &= (x_1, x_2, x_4, x_3), \\ \sigma_3(x_1, x_2, x_3, x_4) &= (x_3, x_4, x_1, x_2). \end{aligned}$$

The group generated by them is the dihedral group of order eight \mathfrak{D}_4 , that defines the rotations and the reflexions of an affine plane preserving a regular polygon with four vertices. Let $\sigma_5 = \sigma_1\sigma_3$. In terms of the involutions (3.2), \mathfrak{D}_4 has the following presentation

$$\mathfrak{D}_4 = \langle \sigma_3, \sigma_5 \mid \sigma_3^2 = \sigma_5^4 = 1, \sigma_3\sigma_5\sigma_3 = \sigma_5^{-1} \rangle.$$

It is easy to deduce from this description that \mathfrak{D}_4 may be realized as a semidirect product $\langle \sigma_3 \rangle \ltimes \langle \sigma_5 \rangle$. The element σ_5 generates a cyclic group of order four that describes the rotations of the plane through an angle of $\frac{\pi}{2}$. We indicate this group by \mathfrak{C}_4 . The element σ_3 (one of the four reflections of the plane) generates a cyclic group of order two and it conjugates elements of \mathfrak{C}_4 into their inverses. The center $Z(\mathfrak{D}_4)$ of \mathfrak{D}_4 is generated by $\sigma_5^2 (= \sigma_1\sigma_2)$.

The exact sequence (3.1) becomes a sequence of \mathfrak{D}_4 -modules. In the rest of this paragraph we provide a description of the map e , restricted to each isotypic space for the action of the characters of \mathfrak{D}_4 .

It is a standard fact that each element of \mathfrak{D}_4 can be written uniquely either in the form σ_5^k or in the form $\sigma_3\sigma_5^k$ for $0 \leq k \leq 3$. We define $\sigma_4 = \sigma_5^2$, $\sigma_6 = \sigma_5^3$, $\sigma_7 = \sigma_3\sigma_5^2$, whereas we have that $\sigma_1 = \sigma_3\sigma_5^3$ and $\sigma_2 = \sigma_3\sigma_5$.

The group \mathfrak{D}_4 has five conjugacy classes and therefore five characters. Each of the four 1-dimensional irreducible characters χ_i , $i = 1, \dots, 4$, factorizes through the center $Z(\mathfrak{D}_4)$ (i.e. $\chi_i(\sigma_5^2) = 1$). These characters are uniquely characterized by their values on σ_1 and σ_3 . The fifth character χ_5 is obtained by inducing a 1-dimensional irreducible representation of \mathfrak{C}_4 . Let χ be its character. We choose for it the following description

$$\chi : \mathfrak{C}_4 \rightarrow \mathbb{C}^*, \quad \chi(\sigma_5^k) = \exp\left(\frac{\pi i k}{2}\right), \quad 0 \leq k \leq 3.$$

The element σ_3 acts on χ as $(\sigma_3\chi)(\sigma_5^k) = \chi(\sigma_3\sigma_5^k\sigma_3)$. Because this action does not stabilize the character χ ($\sigma_3\chi \neq \chi$), one deduces that the identity is the only irreducible representation of the isotropy group of χ . The definition of the character χ_5 of the corresponding irreducible 2-dimensional representation of \mathfrak{D}_4 follows then by a standard induction argument. Explicitely, we have

$$\chi_5 = \text{Ind}_{\mathfrak{C}_4}^{\mathfrak{D}_4}(\chi); \quad \chi_5(\sigma_5^k) = 2 \cos \frac{\pi k}{2}, \quad \chi_5(\sigma_3\sigma_5^k) = 0, \quad 0 \leq k \leq 3.$$

The following table resumes the above description.

(3.3)

characts/elts	$\chi_1 = id$	χ_2	χ_3	χ_4	χ_5
id	1	1	1	1	2
$\sigma_3\sigma_5^3$	1	-1	1	-1	0
$\sigma_3\sigma_5$	1	-1	1	-1	0
σ_3	1	1	-1	-1	0
σ_5^2	1	1	1	1	-2
σ_5	1	-1	-1	1	0
σ_5^3	1	-1	-1	1	0
$\sigma_3\sigma_5^2$	1	1	-1	-1	0

Next, we study the behavior of the evaluation map e on the isotypic subspaces. Our goal is the determination of the rank of the vector spaces $\ker(e) \simeq H^0(\mathbf{P}^4, \mathfrak{I}_S \mathcal{O}(5))$ and $\operatorname{coker}(e) \simeq H^1(\mathbf{P}^4, \mathfrak{I}_S \mathcal{O}(5))$, together with the description of their \mathfrak{D}_4 -module structure. This will be accomplished by a direct computation of the rank of their isotypic subspaces.

As a linear map of vector spaces, e is described by a rectangular matrix of size 126×120 . Because e is \mathfrak{D}_4 -linear, it carries the isotypic subspace for a certain irreducible representation to the subspace relative to the same representation within the vector space of the singular points.

The space $H^0(\mathbf{P}^4, \mathcal{O}(5))$ is generated by monomials of degree five in the homogeneous variables x_0, \dots, x_4 . On each of the five affine charts that cover the projective space \mathbf{P}^4 , one writes down the 126 monomials in the corresponding four affine coordinates. The group \mathfrak{D}_4 acts on them by switching the coordinates x_i (*cf.* (3.2)). The matrix that describes this action has size 126×126 . One may consider its projection onto the eigenspace associated to each irreducible character of \mathfrak{D}_4 .

Let $\rho : \mathfrak{D}_4 \rightarrow \mathbf{GL}(H^0(\mathbf{P}^4, \mathcal{O}(5)))$ denote the linear representation of \mathfrak{D}_4 , with character ϕ and let V_i ($i = 1, \dots, 5$) be the isotypic spaces corresponding to the irreducible characters χ_i of degrees n_i . The projection formula onto the i -th isotypic space, reads as

$$p_i = \frac{n_i}{8} \sum_{\sigma \in \mathfrak{D}_4} \chi_i(\sigma) \rho(\sigma).$$

One writes down the canonical decomposition of $H^0(\mathbf{P}^4, \mathcal{O}(5))$ into irreducible representations, each of which appears with the corresponding multiplicity

$$(3.4) \quad (\phi | \chi_i) = \langle \phi, \chi_i \rangle = \frac{1}{8} \sum_{\sigma \in \mathfrak{D}_4} \phi(\sigma) \chi_i(\sigma) = \frac{1}{8} \sum_{\sigma \in \mathfrak{D}_4} \chi_i(\sigma) \operatorname{Tr}(\rho(\sigma)).$$

The computation of the multiplicities in (3.4) can be easily done by hands. The idea is to keep track of the number of monomials that are invariant under the action of the group \mathfrak{D}_4 . The explicit description of the eigenvectors generating each isotypic subspace was obtained with the help of a computer. One gets the following decomposition

Proposition 3.1. *The canonical decomposition of $H^0(\mathbf{P}^4, \mathcal{O}(5))$ as a \mathfrak{D}_4 -module is*

$$H^0(\mathbf{P}^4, \mathcal{O}(5)) = 27V_1 \oplus 9V_2 \oplus 23V_3 \oplus 7V_4 \oplus 30V_5.$$

We denote by V_i the irreducible representation associated to the character χ_i . The number in front of each eigenspace corresponds to its degree.

To get a similar decomposition for the vector space of the singular points, one has to determine their coordinates explicitly and understand the orbits for the action of \mathfrak{D}_4 on them. We accomplished this part using some invariant properties of the Chebyshev polynomial P_5 . We review them briefly in the following.

Let think of $P_5(x_1, x_2)$ as a map $P_5 : \mathbb{C}_{(x_1, x_2)}^2 \rightarrow \mathbb{C}$. We consider a set of coordinates $\{u_1, u_2, u_3\}$ satisfying the equation $u_1 + u_2 + u_3 = 0$. These coordinates are related to x_1 and x_2 by the map h

$$(3.5) \quad \begin{aligned} h : \mathbb{C}_{(u_1, u_2, u_3)}^2 &\rightarrow \mathbb{C}_{(x_1, x_2)}^2, \\ x_1 &= \exp(-2\pi i u_1) + \exp(-2\pi i u_2) + \exp(-2\pi i u_3) \\ x_2 &= \exp(-2\pi i(u_1 + u_2)) + \exp(-2\pi i(u_1 + u_3)) + \exp(-2\pi i(u_2 + u_3)). \end{aligned}$$

The function h plays for the generalized Chebyshev polynomial a role analogous to that played by the cosine for the classical Chebyshev polynomial in one variable. Namely, h is invariant under the affine Weyl group associated to the root system A_2 . We introduce a second set of coordinates $\{u_4, u_5, u_6\}$ satisfying relations similar to those described by the first set. The set $\{u_4, u_5, u_6\}$ is related to the coordinates x_4, x_5 via equations analogous to (3.5). Then

$$(3.6) \quad P_5(h(u_1, u_2, u_3)) - P_5(h(u_4, u_5, u_6)) = 0, \quad u_1 + u_2 + u_3 = 0 = u_4 + u_5 + u_6$$

describes an affine covering Y of the threefold X . To make as simple as possible the notations, we rename the complex variables as

$$y_1 = \exp(-2\pi i u_1), \quad y_2 = \exp(-2\pi i u_2), \quad y_4 = \exp(-2\pi i u_4), \quad y_5 = \exp(-2\pi i u_5).$$

Then, Y is defined in terms of the variables y_i , as the set of solutions of

$$(3.7) \quad \begin{aligned} y_1^5 + y_2^5 + y_3^5 + y_1^{-5} + y_2^{-5} + y_3^{-5} &= y_4^5 + y_5^5 + y_6^5 + y_4^{-5} + y_5^{-5} + y_6^{-5} \\ y_3 &= (y_1 y_2)^{-1}, \quad y_6 = (y_4 y_5)^{-1}. \end{aligned}$$

Namely, Y is defined in terms of the polynomials f introduced in (2.1) as the sum of two Dickson polynomials of the first kind.

The algebraic description of the covering $\pi : Y \rightarrow X$ is induced by the restriction of the map

$$(3.8) \quad \begin{aligned} \pi^* : \mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6] &\rightarrow \mathbb{C}[y_1, y_2, y_3, y_4, y_5, y_6] \\ \pi^*(x_1, x_2, x_3, x_4, x_5, x_6) &= (y_1 + y_2 + y_3, y_1 y_2 + y_1 y_3 + y_2 y_3, y_1 y_2 y_3, \\ &\quad y_4 + y_5 + y_6, y_4 y_5 + y_4 y_6 + y_5 y_6, y_4 y_5 y_6) \end{aligned}$$

to the four dimensional affine space $\mathbf{A}^4 = \{x_3 = 1 = x_6\}$. One obtains

$$\pi^* : \mathbb{C}[x_1, x_2, x_4, x_5] / (f_5(x_1, x_2, x_4, x_5)) \rightarrow \mathbb{C}[y_1, y_2, \dots, y_6] / (y_1 y_2 y_3 - 1, y_4 y_5 y_6 - 1)$$

where f_5 was defined in (2.5). If $P = (y_1, \dots, y_6)$ is a point in Y , the definition of the map $\pi : Y \rightarrow X$ is given by

$$\begin{aligned} \pi(P) &= \pi(y_1, \dots, y_6) = \\ &= (y_1 + y_2 + \frac{1}{y_1 y_2}, y_1 y_2 + \frac{1}{y_1} + \frac{1}{y_2}, y_4 + y_5 + \frac{1}{y_4 y_5}, y_4 y_5 + \frac{1}{y_4} + \frac{1}{y_5}). \end{aligned}$$

The quotient X is the variety of the orbits for the action of $(\mathfrak{S}_3)^2$ on Y (\mathfrak{S}_i = symmetric group in i variables). Each of the two factors \mathfrak{S}_3 acts on the sets of coordinates $\{y_1, y_2, y_3\}$ *resp.* $\{y_4, y_5, y_6\}$. The fiber of π over a point $Q = (x_1, x_2, x_4, x_5) \in X$, coincides with its (closed) orbit and it is given by the set (*i.e.* closed zero scheme) of the ordered pairs $(y_1, y_2, y_3, y_4, y_5, y_6)$ of points of Y (at most 36 distinct, multiple pairs are possible), where y_1, y_2, y_3 *resp.* y_4, y_5, y_6 range among the permutations of the zeroes of the equation

$$(3.9) \quad T^3 - x_1 T^2 + x_2 T - 1 = 0; \quad \text{for:} \quad \begin{aligned} x_1 &= y_1 + y_2 + y_3 \\ x_2 &= y_1 y_2 + y_1 y_3 + y_2 y_3 \\ y_1 y_2 y_3 &= 1 \end{aligned}$$

resp.

$$(3.10) \quad T^3 - x_4 T^2 + x_5 T - 1 = 0; \quad \text{for:} \quad \begin{aligned} x_4 &= y_4 + y_5 + y_6 \\ x_5 &= y_4 y_5 + y_4 y_6 + y_5 y_6 \\ y_4 y_5 y_6 &= 1. \end{aligned}$$

The coordinate ring of X coincides with the \mathfrak{S}_3^2 -invariants of the coordinate ring of Y : *i.e.* the homomorphism (3.8) is described by the embedding $A(X) = A(Y)^{\mathfrak{S}_3^2} \hookrightarrow A(Y)$. This agrees with the fact that f_5 is a linear combination of elementary symmetric functions in the two separate sets of variables $\{x_1, x_2\}$ and $\{x_4, x_5\}$.

Using the equality $\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$ one gets

$$P_5(h(u_1, u_2, u_3)) = 2 \cos(10\pi u_1) + 2 \cos(10\pi u_2) + 2 \cos(10\pi u_3).$$

By a direct computation (*cf.* [7]), it is easy to see that if $(\zeta_1, \zeta_2, \zeta_3)$ is a critical point of the compositum $P_5(h(u_1, u_2, u_3))$, then $(\zeta_i)^{30} = 1$, $i = 1, \dots, 3$. The values taken at the critical points are 6, -2, -3. In other words, the 6-tuple $(\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6)$ is a singular point of Y if (and only if) it satisfies the conditions

$$(3.11) \quad \begin{aligned} \zeta_i &= e^{-\pi i \frac{\alpha_i}{15}}, \quad \alpha_i \in \mathbb{Z}, \quad i = 1, 2, 4, 5 \\ \zeta_3 &= (\zeta_1 \zeta_2)^{-1}, \quad \zeta_6 = (\zeta_4 \zeta_5)^{-1}. \end{aligned}$$

Proposition 3.2. *The singular locus of the quasi-projective variety Y consists of 8750 double points.*

Proof. If P is a point in the singular locus of the variety Y (*cf.* (3.7)), then the coordinates of P must satisfy the conditions (3.11). This implies that for $P = (\zeta_1, \dots, \zeta_6) \in \text{Sing}(Y)$, $\zeta_i^5 = e^{-\pi i \frac{\alpha_i}{3}}$ for $i = 1, 2, 4, 5$ and $\alpha_i = 1, \dots, 5$. Therefore, the singular locus of

Y is described by the set of points $(e^{-\pi i \frac{\alpha_1}{3}}, e^{-\pi i \frac{\alpha_2}{3}}, e^{-\pi i \frac{(\alpha_1 + \alpha_2)}{3}}, e^{-\pi i \frac{\alpha_4}{3}}, e^{-\pi i \frac{\alpha_5}{3}}, e^{-\pi i \frac{(\alpha_4 + \alpha_5)}{3}})$ in the singular locus of the variety

(3.12)

$$x_1 + x_2 + x_1 x_2 + x_1^{-1} + x_2^{-1} + (x_1 x_2)^{-1} = x_4 + x_5 + x_4 x_5 + x_4^{-1} + x_5^{-1} + (x_4 x_5)^{-1}.$$

The 4-tuples $(\alpha_1, \alpha_2, \alpha_4, \alpha_5)$ are chosen among the set of solutions of

$$\cos(\pi \frac{\alpha_1}{3}) + \cos(\pi \frac{\alpha_2}{3}) + \cos(\pi \frac{\alpha_1 + \alpha_2}{3}) = \cos(\pi \frac{\alpha_4}{3}) + \cos(\pi \frac{\alpha_5}{3}) + \cos(\pi \frac{\alpha_4 + \alpha_5}{3})$$

under the condition $0 \leq \alpha_i \leq 5$. By a direct computation one obtains the following locus

$$\begin{aligned} T = \{ & (1, 1, 1, 1, 1, 1), (1, -1, -1, 1, -1, -1), (1, -1, -1, -1, 1, -1), (1, -1, -1, -1, \\ & -1, 1), (e^{-2\pi i/3}, e^{-2\pi i/3}, e^{-2\pi i/3}, e^{-2\pi i/3}, e^{-2\pi i/3}, e^{-2\pi i/3}), (e^{-2\pi i/3}, e^{-2\pi i/3}, e^{-2\pi i/3}, e^{2\pi i/3}, \\ & e^{2\pi i/3}, e^{2\pi i/3}), (-1, 1, -1, 1, -1, -1), (-1, 1, -1, -1, 1, -1), (-1, 1, -1, -1, -1, 1), \\ & (-1, -1, 1, 1, -1, -1), (-1, -1, 1, -1, 1, -1), (-1, -1, 1, -1, -1, 1), \\ & (e^{2\pi i/3}, e^{2\pi i/3}, e^{2\pi i/3}, e^{-2\pi i/3}, e^{-2\pi i/3}, e^{-2\pi i/3}), (e^{2\pi i/3}, e^{2\pi i/3}, e^{2\pi i/3}, e^{2\pi i/3}, e^{2\pi i/3}, e^{2\pi i/3}) \}. \end{aligned}$$

Hence, to each of the 14 points in T correspond 5^4 points solutions of (3.7). In this way, one counts globally $5^4 \times 14 = 8750$ singular points for Y . One can direct verify that they are all rational double points. \square

The semi-direct product $G := (\mathfrak{S}_3^2 \rtimes \mathfrak{S}_2) \rtimes \mu_5^4$ acts on Y ($\mu_5 =$ fifth roots of unity). The action of \mathfrak{S}_2 interchanges the two sets of coordinates $\{y_1, y_2, y_3\}$ and $\{y_4, y_5, y_6\}$ and it is preserved by the morphism π . It corresponds to the action of the involution σ_3 on X (cf. (3.2)). The abelian subgroup μ_5^4 acts freely on Y and on its singular set. The action of $\mathfrak{S}_3 \times \mathfrak{S}_3$ has instead stabilizers. If $S' = \text{Sing}(Y)$, then the action of μ_5^4 on S' determines the 14 orbits of points described by the set T . In other words, the vector space $H^0(S', \mathcal{O}_{S'})$, inherits a structure of μ_5^4 -module. The related μ_5^4 -representation is a direct sum of 14 copies of the regular representation $H^0(S', \mathcal{O}_{S'}) = \bigoplus_{14} \mathbb{C}[\mu_5^4]$.

We divide the 8750 singular points in S' in accord with the action that $\mathfrak{S}_3 \times \mathfrak{S}_3$ has on them. We are mainly interested in understanding the unramified part of this action (*i.e.* to focus on the points not contained in the ramification locus (3.12) of π). In fact, the unramified points define a set (*i.e.* zero scheme) on which \mathfrak{S}_3^2 acts freely: the quotient space for this action determines the (non singular) zero-scheme S of X -rational double points. Each point in T has a non-trivial stabilizer for the action of $\mathfrak{S}_3 \times \mathfrak{S}_3$ and therefore the image by π of points within their μ_5^4 -orbits determines the singular locus of X .

In the following part we give some details on how we accomplished these calculations: the numbers were checked with the help of a computer. Within the following five orbits

of T

$$\{(1, 1, 1, 1, 1, 1), (e^{-2\pi i/3}, e^{-2\pi i/3}, e^{-2\pi i/3}, e^{-2\pi i/3}, e^{-2\pi i/3}, e^{-2\pi i/3}), (e^{-2\pi i/3}, e^{-2\pi i/3}, e^{-2\pi i/3}, e^{2\pi i/3}, e^{2\pi i/3}, e^{2\pi i/3}), (e^{2\pi i/3}, e^{2\pi i/3}, e^{2\pi i/3}, e^{-2\pi i/3}, e^{-2\pi i/3}, e^{-2\pi i/3}), (e^{2\pi i/3}, e^{2\pi i/3}, e^{2\pi i/3}, e^{2\pi i/3}, e^{2\pi i/3}, e^{2\pi i/3})\}$$

one counts $5 \cdot (12)^2 = 720$ $(\mathfrak{S}_3 \times \mathfrak{S}_3)$ -unramified Y -points. More precisely, each of these orbits contains representatives of $144/36 = 4$ $(\mathfrak{S}_3 \times \mathfrak{S}_3)$ -(unramified) orbits. The complete orbits determine $4 \times 5 = 20$ X -singular points globally. In the remaining 9 orbits of T there are $9 \cdot (20)^2 = 3600$ $(\mathfrak{S}_3 \times \mathfrak{S}_3)$ -unramified Y -points subdivided in the following way. Four among the 9 orbits contain 8 representatives of $(\mathfrak{S}_3 \times \mathfrak{S}_3)$ -(unramified) orbits each. The (complete) orbits associated to these representatives (that are distributed in different μ_5^4 -orbits of the set T) determine 32 X -singular points globally. Four further orbits (among the 9) contain 16 representatives of $(\mathfrak{S}_3 \times \mathfrak{S}_3)$ -(unramified) orbits each. The associated complete orbits determine 64 singular points on X . Finally, the last μ_5^4 -orbit contains 4 representatives of $(\mathfrak{S}_3 \times \mathfrak{S}_3)$ -(unramified) orbits. Hence one counts 4 singular points on X . The total number of $(\mathfrak{S}_3 \times \mathfrak{S}_3)$ -unramified Y -points is $720 + 3600 = 4320$. They correspond to $4320/36 = 120$ singular points of S .

Using the explicit description of these points in logarithmic (u) -coordinates (*cf.* (3.5)), one determines the \mathfrak{D}_4 -action on them. First, one writes down the character of the representation $\tilde{\rho} : \mathfrak{D}_4 \rightarrow \mathbf{GL}(H^0(S, \mathcal{O}_S))$ as a trace of the automorphism given by the action of \mathfrak{D}_4 on the 120 $(\mathfrak{S}_3 \times \mathfrak{S}_3)$ -unramified points in the 14 orbits of T . Then, one computes the multiplicities as in (3.4) of each irreducible representation associated to $\tilde{\rho}$. This can be easily done with the help of a computer. We obtained the following canonical decomposition

Proposition 3.3. *The canonical decomposition of the \mathfrak{D}_4 -module $H^0(S, \mathcal{O}_S)$ is*

$$H^0(S, \mathcal{O}_S) = 27U_1 \oplus 13U_2 \oplus 17U_3 \oplus 7U_4 \oplus 28U_5.$$

Here U_i is the irreducible representation associated to the character χ_i and the number in front of it corresponds to its multiplicity.

For simplicity, we re-number the projective coordinates used in (3.8) as $\{x_0, x_1, x_2, x_3, x_4\}$.

Next, we describe the decomposition in eigenspaces of the 25-dimensional subvector space $K \subset \ker(e)$, generated by the linear combinations

$$\sum_{i=0}^4 L(x_0, \dots, x_4) \cdot \frac{\partial F_5}{\partial x_i}(x_0, \dots, x_4).$$

This is obtained using techniques similar to the ones already described. Firstly, one writes the character associated to the representation $\mathfrak{D}_4 \rightarrow GL(K)$ and then one computes the multiplicities of each irreducible representation associated to it. The action of \mathfrak{D}_4 on the five partial derivatives and on the linear functions L is given by

$$\sigma_i\left(\frac{\partial F_5}{\partial x_j}\right) = \sum_j a_{ij} \sigma_i\left(\frac{\partial F_5}{\partial x_j}\right), \quad \text{and} \quad \sigma_i(x_j) = \sum_j b_{ij} \sigma_i(x_j).$$

The character associated to this representation is the tensor product of the two characters $\chi(\sigma_i) = \text{tr}(a_{ij}(\sigma_i))$ and $\chi'(\sigma_i) = \text{tr}(b_{ij}(\sigma_i))$. Straightforward computations determine the following table

characts/elts	χ	χ'
id	5	5
σ_1	3	3
σ_2	3	3
σ_3	-1	1
σ_4	1	1
σ_5	-1	1
σ_6	-1	1
σ_7	-1	1

The values taken by the tensor product $\chi \otimes \chi'$ (*i.e.* the character of the 25-dimensional representation) are readable from the above table.

The scalar products

$$m_i = \langle \chi \otimes \chi', \chi_i \rangle = \frac{1}{8} \sum_{\sigma_j \in \mathfrak{D}_4} (\chi \otimes \chi')(\sigma_j) \cdot \chi_i(\sigma_j)$$

determine the multiplicities of each irreducible representation. One obtains

$$K = 5K_1 \oplus K_2 \oplus 6K_3 \oplus K_4 \oplus 6K_5.$$

It remains to describe the canonical decomposition and the dimension of $\ker(e)$ itself. This can be done by evaluating the eigenvectors that define the eigenspaces associated to each irreducible character at the 120 double points. For each of these five maps we computed its kernel and rank and obtained the following canonical decomposition

Proposition 3.4. $H^0(\mathbf{P}^4, \mathfrak{I}_S \mathcal{O}(5)) \simeq \ker(e) = 5K_1 \oplus K_2 \oplus (6K_3 \oplus \mathbb{C}v_3) \oplus K_4 \oplus 6K_5.$

The following corollary resumes what we obtained.

Corollary 3.5. *With the above notations, one has*

$$\text{rk } H^0(\mathbf{P}^4, \mathfrak{I}_S \mathcal{O}(5)) = 26, \quad \text{rk } H^1(\mathbf{P}^4, \mathfrak{I}_S \mathcal{O}(5)) = 20.$$

In particular, the canonical decomposition of the \mathfrak{D}_4 -module $H^1(\mathbf{P}^4, \mathfrak{I}_S \mathcal{O}(5))$ is

$$H^1(\mathbf{P}^4, \mathfrak{I}_S \mathcal{O}(5)) = 5U'_1 \oplus 5U'_2 + U'_3 + U'_4 + 4U'_5$$

where U'_i is the irreducible representation associated to the character χ_i .

Remark 3.6.

For the description of the eigenvector $v_3 \in \ker(e) - K$ one looks at the complement of $6K_3$ in the 23-dimensional isotypic space $V_3 \subset H^0(\mathbf{P}^4, \mathcal{O}(5))$ associated to the character χ_3 . One searches for a linear combination of the 17 elements generating the

complement of $6K_3$ in the space $23V_3$. The following χ_3 -invariant affine polynomial satisfies these conditions and it was found with the help of a computer.

$$\begin{aligned} h(x_1, \dots, x_4) = & -x_1^2x_3 - x_2^2x_3 + x_1x_3^2 + x_2x_3^2 - x_1^2x_4 - x_2^2x_4 + x_1x_4^2 + x_2x_4^2 + (-x_1^4x_3 - \\ & - x_2^4x_3 + x_1x_3^4 + x_2x_3^4 - x_1^4x_4 - x_2^4x_4 + x_1x_4^4 + x_2x_4^4) + [-x_1x_2x_3 - x_1x_2x_4 + x_1x_3x_4 + \\ & + x_2x_3x_4 + (-x_1^2x_2^2x_3 - x_1^2x_2^2x_4 + x_1x_3^2x_4^2 + x_2x_3^2x_4^2)] + [-x_1^3x_3 - x_2^3x_3 + x_1x_3^3 + x_2x_3^3 - x_1^3x_4 - \\ & - x_2^3x_4 + x_1x_4^3 + x_2x_4^3 - (x_1^2x_2x_3 - x_1x_2^2x_3 - x_1^2x_2x_4 - x_1x_2^2x_4 + x_1x_3^2x_4 + x_2x_3^2x_4 + \\ & + x_1x_3x_4^2 + x_2x_3x_4^2)] \end{aligned}$$

□

From these computations one may deduce the Hodge structure on $H^*(\tilde{X}, \mathbb{C})$. We are only concerned with the knowledge of $H^2(\tilde{X})$ and $H^3(\tilde{X})$ as one knows that $h^1(\tilde{X}) = 0$. The following complex of sheaves in \mathbf{P}^4

$$(3.13) \quad \mathcal{E}^\cdot : \quad 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \Omega_{\mathbf{P}^4}^3(X) \xrightarrow{d} \mathfrak{I}_S \omega_{\mathbf{P}^4}(2X)$$

induces in hypercohomology the isomorphism

$$F^2 H^i(\tilde{X}) \simeq H^{i+1}(\mathbf{P}^4, \mathcal{E}^\cdot).$$

For a proof of this statement we refer to [20]. The only interesting result is related to the index $i = 3$. Namely one gets

$$(3.14) \quad F^2 H^3(\tilde{X}, \mathbb{C}) \simeq \frac{H^0(\mathbf{P}^4, \mathfrak{I}_S \omega_{\mathbf{P}^4}(2X))}{\text{Image}(H^0(\mathbf{P}^4, \Omega^3(X)) \rightarrow H^0(\mathbf{P}^4, \mathfrak{I}_S \omega_{\mathbf{P}^4}(2X)))}.$$

Hence

Corollary 3.7. *Under the same notations*

$$h^{3,0}(\tilde{X}) = 1 = h^{2,1}(\tilde{X}).$$

Proof. It follows from the above isomorphism and from the fact that $h^0(\mathbf{P}^4, \Omega^3(X)) = 24$ (independent of the number of the singular points). For smooth, quintic hypersurfaces in \mathbf{P}^4 one knows that $h^{3,0} = 1$. □

Remark 3.8.

The χ_3 -invariant polynomial h defined in Remark 3.6 is a generator of the space $H^{2,1}(\tilde{X})$. From (3.13) and (3.14) one obtains the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(\mathbf{P}^4, \Omega^3(X)) & \xrightarrow{d} & H^0(\mathbf{P}^4, \mathfrak{I}_S \omega_{\mathbf{P}^4}(2X)) & \xrightarrow{\beta} & F^2 H^3(\tilde{X}, \mathbb{C}) \longrightarrow 0 \\ & & & & \uparrow \simeq \psi & & \\ & & & & H^0(\mathbf{P}^4, \mathfrak{I}_S \mathcal{O}(5)) & & \end{array}$$

The isomorphism ψ is induced by the isomorphism of sheaves $\omega_{\mathbf{P}^4} \simeq \mathcal{O}_{\mathbf{P}^4}(-5)$. The map β is deduced from the blow-up of X along S and by the Poincaré residue. The composite of these maps is described by

$$P(x_0, \dots, x_4) \mapsto \frac{P(x_0, \dots, x_4) \cdot \sum_{i=0}^4 (-1)^i dx_0 \wedge \dots \wedge \hat{dx}_i \wedge \dots \wedge dx_4}{F_5^2(x_0, \dots, x_4)}.$$

Here $P(x_0, \dots, x_4) \in H^0(\mathbf{P}^4, \mathcal{I}_S \mathcal{O}(5))$ is a homogeneous polynomial of degree 5 that vanishes at each point of S and $F_5(x_0, \dots, x_4)$ is the homogeneous polynomial that defines X .

It is worth to remark that if one chooses $P(x_0, \dots, x_4) = F_5(x_0, \dots, x_4)$, then its image by means of $\beta \cdot \psi$ gives a generator of the subvector space $H^0(\tilde{X}, \omega_{\tilde{X}})$ of the holomorphic 3-forms on \tilde{X} . In other words, the Kähler differential from \mathbf{P}^4 via Poincaré residue pulls-back to a regular 3-form on \tilde{X} . If instead one chooses $P = h$, then it follows from Remark 3.6 that the image of it via $\beta \cdot \psi$ generates the $(2, 1)$ -part of the $H^3(\tilde{X})$.

Finally, let $y_i = \frac{x_i}{x_0}$ (a similar argument works in each of the four affine charts), then the differential form $\omega = dy_1 \wedge dy_2 \wedge dy_3 \wedge dy_4$ is χ_2 -invariant as $\sigma_1(\omega) = \sigma_2(\omega) = -\omega$ and $\sigma_3(\omega) = \omega$ (cf. the character table description). Therefore, the image of any χ_3 -invariant polynomial (e.g. F_5 and h) by $\beta \cdot \psi$, becomes $\chi_2 \chi_3 (= \chi_4)$ -invariant. Because $\chi_4(\sigma_1) = -1 = \chi_4(\sigma_3)$, it follows that one cannot split the rank 4 motive associated to $F^2 H^3(\tilde{X})$ by the action of any subgroup of \mathfrak{D}_4 on \tilde{X} . □

We recall the well known fact

Lemma 3.9. *Let \tilde{X} be an hypersurface of degree five in \mathbf{P}^4 with 120 nodes (i.e. rational double points). Then, the Euler characteristic of a desingularization \tilde{X} of \tilde{X} obtained by blowing up the nodes is given by the formula*

$$\chi(\tilde{X}) = \chi(X_\eta) + 4 \cdot 120 = -200 + 4 \cdot 120 = 280.$$

Here \tilde{X}_η is the generic fiber of a Lefschetz fibration of degree five hypersurfaces in \mathbf{P}^4 with \tilde{X} as a special fiber.

Proof. The vanishing cycle exact sequence applied to a Lefschetz pencil of quintic hypersurfaces in \mathbf{P}^4 with degenerating fiber \tilde{X} and generic fiber \tilde{X}_η , reads as

$$0 \rightarrow H^3(\tilde{X}, \mathbb{Q}) \rightarrow H^3(\tilde{X}_\eta, \mathbb{Q}) \rightarrow V \rightarrow H^4(\tilde{X}, \mathbb{Q}) \rightarrow H^4(\tilde{X}_\eta, \mathbb{Q}) \rightarrow 0$$

$$H^i(\tilde{X}, \mathbb{Q}) \simeq H^i(\tilde{X}_\eta, \mathbb{Q}), \quad \text{for } i \neq 3, 4.$$

Here, V is the \mathbb{Q} -vector space of vanishing cycles. Because \tilde{X} has only double points, the rank of V equals the number of double points of \tilde{X} . Hence one gets

$$(3.15) \quad \chi(\tilde{X}) = \chi(\tilde{X}_\eta) + 120.$$

The Euler charactersitic of \tilde{X}_η (a non-singular hypersurface of degree 5 in \mathbf{P}^4) is given by integrating its third Chern class against \tilde{X}_η and then by interpreting the integral

using Stokes theorem

$$\chi(\bar{X}_\eta) = \int_{\bar{X}_\eta} c_3(\bar{X}_\eta) = \int_{\mathbf{P}^4} [5J] \wedge c_3(\mathcal{O}_{\mathbf{P}^4}).$$

Here, $J = c_1(\mathcal{O}_{\mathbf{P}^4}(1))$. It is well known that for degree q smooth hypersurfaces in $\mathbf{P}_{\mathbb{C}}^n$ the r -th Chern class is given by

$$c_r = \left[\sum_{k=0}^r \binom{n+1}{k} (-q)^{r-k} \right] J^r.$$

For degree 5 hypersurfaces in \mathbf{P}^4 one gets $\chi(\bar{X}_\eta) = -200$. Using Lefschetz hyperplane theorem, one may deduce that $h^2(\bar{X}_\eta) = h^{1,1}(\bar{X}_\eta) = 1$ and $h^1(\bar{X}_\eta) = 0$. Hence, $h^{2,1}(\bar{X}_\eta) = 101$ as $h^{3,0}(\bar{X}_\eta) = 1$. In turn, this result implies $\chi(\bar{X}) = 2 + h^2(\bar{X}) - h^3(\bar{X}) + h^4(\bar{X}) = 2 + 1 - h^3(\bar{X}) + h^4(\bar{X}) = 3 - h^3(\bar{X}) + h^4(\bar{X}) = -80$. Hence $h^3(\bar{X}) - h^4(\bar{X}) = 83$. The Leray spectral sequence for blow-up of double points $\tilde{X} \rightarrow \bar{X}$ gives the following exact sequences

$$(3.16) \quad \begin{aligned} 0 \rightarrow H^2(\bar{X}, \mathbb{Q}) \rightarrow H^2(\tilde{X}, \mathbb{Q}) \rightarrow H^2(E, \mathbb{Q}) \rightarrow H^3(\bar{X}, \mathbb{Q}) \rightarrow H^3(\tilde{X}, \mathbb{Q}) \rightarrow 0 \\ 0 \rightarrow H^4(\bar{X}, \mathbb{Q}) \rightarrow H^4(\tilde{X}, \mathbb{Q}) \rightarrow H^4(E, \mathbb{Q}) \rightarrow 0 \end{aligned}$$

$$H^i(\bar{X}, \mathbb{Q}) = H^i(\tilde{X}, \mathbb{Q}), \quad \text{for } i \neq 2, 3, 4.$$

Here E is the exceptional fiber, *i.e.* a union of $120 \mathbf{P}^1 \times \mathbf{P}^1$. From this sequence it follows that $\chi(\tilde{X}) = \chi(\bar{X}) + 120 \cdot (\chi(\mathbf{P}^1 \times \mathbf{P}^1) - 1)$. Using (3.15) one concludes that

$$\chi(\tilde{X}) = \chi(\bar{X}_\eta) + 120 \cdot \chi(\mathbf{P}^1 \times \mathbf{P}^1) = -200 + 4 \cdot 120 = 280.$$

□

As an immediate consequence we have

Corollary 3.10. $h^{1,1}(\tilde{X}) = 141 = h^2(\tilde{X})$, $h^3(\bar{X}) = 104$, and $h^4(\bar{X}) = 21$.

Proof. The equality $h^2(\tilde{X}) = h^{1,1}(\tilde{X})$ follows from (3.16) and the fact that the rank-one vector space $H^2(\tilde{X}, \mathbb{Q})$ is generated by the class of a hyperplane section. The equality $h^2(\tilde{X}) = 141$ is a direct consequence of the above lemma and corollary 3.7 (*i.e.* $h^3(\tilde{X}) = 4$). The equality $h^3(\bar{X}) = 104$ follows from (3.16). Finally, $h^4(\bar{X}) = 21$ is deduced from the difference $h^3(\bar{X}) - h^4(\bar{X}) = 83$ (*cf.* proof of the last lemma). □

4. REDUCTION MOD. p AND COUNTING POINTS.

In this paragraph we describe the reduction of \tilde{X} modulo a prime. We use the same notation as was introduced in the previous paragraphs.

Lemma 4.1. *The threefold \tilde{X} has good reduction outside 2, 3 and 5.*

Proof. In order to prove this, we study the behaviour of the critical points of $P_5(x_1, x_2)$ mod p . In paragraph 2, we observed that $P_5(x_1, x_2)$ has 10 critical points with value -2 , 4 with value -3 and 2 with value 6. We list these points in the following table.

(4.1)	point	# Gal.orbit	field of def.	value
	(w, w)	2	$\mathbb{Q}(w)$	6
	$(w + 1, w + 1)$	2	$\mathbb{Q}(w)$	-2
	$(-\zeta_5, -\zeta_5^{-1})$	4	$\mathbb{Q}(\zeta_5)$	-2
	$(\zeta_5 - \zeta_5^{-1} - 1, \zeta_5^{-1} - \zeta_5 - 1)$	4	$\mathbb{Q}(\zeta_5)$	-2
	$(w\zeta_3, w\zeta_3^{-1})$	4	$\mathbb{Q}(w, \zeta_3)$	-3

One can easily check that these 16 critical points reduce to different points modprimes of $\mathbb{Q}(\zeta_{15})$ above rational primes bigger than 5. One way to verify this is to compute the discriminant with respect to x_2 of the resultant of $\frac{dP_5}{dx_1}$ and $\frac{dP_5}{dx_2}$ with respect to x_1 . This number is only divisible by the primes: 2, 3, 5, 11, 19 and 31, so these are the only primes mod which the critical points can coincide. A finite calculation shows that the critical points reduce to different points mod primes above 11, 19 and 31.

For $p \neq 5$, the resultant of $\frac{dP_5}{dx_1}$ and $\frac{dP_5}{dx_2}$ (say with respect to x_1) is not identically zero. This implies that these polynomials do not have a common factor modulo p . By Bezout's theorem one concludes that $P_5(x_1, x_2)$ mod p has at most 16 critical points. Hence, for $p > 5$ there are precisely 16 critical points: the reductions of the 16 critical points in characteristic zero.

Note that the values -2 , -3 and 6 that P_5 assumes at the critical points remain different when reduced modulo primes bigger than 5. It follows that the number of singular points of the affine singular threefold defined by $P_5(x_1, x_2) = P_5(x_4, x_5)$ does not increase upon reducing modprimes bigger than 5.

At each of the critical points of P_5 one can check that the local expansion of P_5 has a non-degenerate degree-2-part which remains non-degenerate when one reduces it modprimes of $\mathbb{Q}(\zeta_{15})$, above rational primes bigger than 5. It follows that the part of \tilde{X} above the affine part $P_5(x_1, x_2) = P_5(x_4, x_5)$ remains non-singular after the reduction mod $p > 5$.

It remains to verify that \tilde{X} mod p has no singularities at infinity. In fact, the homogenized $P_5(x_1, x_2) - P_5(x_4, x_5)$ has the following form: $x_1^5 + x_2^5 - x_4^5 - x_5^5 + zq(x_1, x_2, x_4, x_5, z)$. If its partial derivatives were all zero at a point where $z = 0$ then, in characteristic $\neq 5$, one would have $x_1 = x_2 = x_4 = x_5 = 0$. \square

We are interested in counting \mathbb{F}_q -rational points on \tilde{X} . This will be accomplished in three steps:

- Determine the number of points on the affine singular part
- Determine how many points are added in the blow-up
- Determine the number of points at infinity.

An efficient way for computing the number of \mathbb{F}_q -rational points on the affinesingular part is as follows. Let x_1 and x_2 run through \mathbb{F}_q , and count how many times P_5 assumes each value. Clearly, the number of points will be the sum of the squares of these values.

In the blow-up each singular point gets replaced by its projectivized tangent cone. Geometrically this is a $\mathbf{P}^1 \times \mathbf{P}^1$, but a priori, there could be different Galois actions on it. In our case it turns out that the rulings of the cone are defined over the same field as the field of definition of the singular point. Therefore, each \mathbb{F}_q -rational singular point contributes $(q+1)^2$ points in the blow-up (instead than 1 point on the singular model). Using the list of critical points of P_5 as in the above table, one finds that the number of \mathbb{F}_q -rational singular points is 120 if $q \equiv 1 \pmod{15}$, 104 if $q \equiv 11 \pmod{15}$, 24 if $q \equiv 4 \pmod{15}$, 8 if $q \equiv 14 \pmod{15}$ and zero otherwise.

The locus at infinity is defined by the homogeneous equation $x_1^5 + x_2^5 - x_4^5 - x_5^5$. One can determine the number of solutions by adding the squares of the number times $x_1^5 + x_2^5$ assumes each value. In this way one gets the number of solutions of the homogeneous equation. One has to subtract 1, and divide by $q-1$ in order to obtain the number of points.

It turns out that for many q one does not need a computer for counting points. One can use the following proposition instead.

Proposition 4.2. *If $q \equiv \pm 2 \pmod{5}$ and $\gcd(q, 30) = 1$, then $\tilde{X}(\mathbb{F}_q) = q^3 + q^2 + q + 1$.*

Proof. Suppose $q \equiv 2$ or $3 \pmod{5}$. In paragraph 1 we showed that the polynomial $P_5(x_1, x_2) = D_5^{(1)} + D_5^{(2)}$, with $D_5^{(1)}$ and $D_5^{(2)}$ Dickson polynomials. It follows from proposition 2.2 that the map $(D_5^{(1)}, D_5^{(2)})$ permutes $\mathbb{F}_q \times \mathbb{F}_q$. Hence $P_5 \pmod{p}$ assumes each value exactly q times and therefore there are $\sum_1^q q^2 = q^3$ solutions to the equation $P_5(x_1, x_2) = P_5(x_4, x_5)$ in F_q . From table 4.1 it follows that there are no critical points defined over \mathbb{F}_q , hence there is no extra contribution coming from the blow-up.

The map $x \mapsto x^5$ permutes the elements of \mathbb{F}_q , hence $x_1^5 + x_2^5$ assumes each value q times. So $x_1^5 + x_2^5 = x_4^5 + x_5^5$ has q^3 solutions in F_q , which correspond to $\frac{q^3-1}{q-1}$ points. In total, one counts globally $q^3 + \frac{q^3-1}{q-1} = q^3 + q^2 + q + 1$ points. \square

Remark 4.3. *The determination of the number of points at infinity also holds for $q \equiv 4 \pmod{5}$.*

5. COMPUTING THE L -FUNCTION.

We are interested in computing the L -function of the Galois action on $H^3(\tilde{X}_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$. We start by recalling its definition. Let I_p be an inertia group at p , and Fr_p a (geometric) Frobenius element of $\text{Gal}_{\mathbb{Q}}$. The L -function is defined as the product

$$L(s) = \prod_{p \text{ prime}} L_p(s)$$

with

$$L_p(s) = \det(\text{Fr}_p \cdot T - \text{Id})^{-1} \Big|_{T=p^{-s}}.$$

Here $(\text{Fr}_p \cdot T - \text{Id})$ acts on the I_p -invariant subspace of $H^3(\tilde{X}_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$.

At the primes p where \tilde{X} has good reduction, $L_p(s)$ can be determined by counting the points on \tilde{X} over various finite fields, and using the Lefschetz trace formula. We

will explain this fact here in more detail. We refer to section 7 for the discussion on how to obtain likely candidates for the $L_p(s)$ at the primes p where \tilde{X} has bad reduction.

Suppose that \tilde{X} has good reduction at a prime p . Then, for $q = p^n$, the Lefschetz trace formula is

$$\tilde{X}(\mathbb{F}_q) = \sum_{i=0}^6 (-1)^i \text{Trace}(\text{Fr}_q | H^i(\tilde{X}_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)).$$

One knows that the trace of the Frobenius on H^0 is 1, and on H^6 is q^3 . The H^1 and H^5 vanish. The H^2 and H^4 are algebraic, and they are related by Poincaré duality. They have dimension 141 (*cf.* corollary 3.10), so there exists an integer k , with $-141 < k < 141$ such that $\text{Trace}(\text{Fr}_q | H^2(\tilde{X}_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)) = kq$ and $\text{Trace}(\text{Fr}_q | H^4(\tilde{X}_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)) = kq^2$. Hence with the the trace formula one obtains

$$(5.1) \quad \text{Trace}(\text{Fr}_q | H^3(\tilde{X}_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)) = 1 + q^3 + k(q + q^2) - \tilde{X}(\mathbb{F}_q).$$

Therefore, the computation of the trace on H^3 is accomplished once one knows k and $\tilde{X}(\mathbb{F}_q)$. In the previous section it was explained how to compute $\tilde{X}(\mathbb{F}_q)$. To determine k one could try to understand the Galois action on the divisors generating H^2 . However, we preferred to proceed differently. In paragraph 3 we proved that $\dim H^3 = 4$ (*cf.* corollary 3.7), and by the Weil conjectures one knows that the eigenvalues of $\text{Fr}_q | H^3$ have absolute value $q^{3/2}$. Hence one gets the inequality

$$(5.2) \quad \left| 1 + q^3 + k(q^2 + q) - \tilde{X}(\mathbb{F}_q) \right| \leq 4q^{3/2}.$$

For q big enough, there will be a unique k that satisfies this inequality. We found that k is uniquely determined if $q > 20$.

The characteristic polynomial of the Fr_q action on H^3 can be expressed in terms of the traces of Fr_q and Fr_{q^2} . Let $a_q = \text{Trace}(\text{Fr}_q | H^3)$, then the characteristic polynomial is

$$T^4 - a_q T^3 + \frac{1}{2}(a_q^2 - a_{q^2})T^2 - q^3 a_q T + q^6,$$

and for $q = p$ the local L -factor becomes

$$L_p(s) = \frac{1}{1 - a_p p^{-s} + \frac{1}{2}(a_p^2 - a_{p^2})p^{-2s} - a_p p^{3-3s} + p^{6-4s}}.$$

This explains how to compute L_p for $p > 20$.

For $7 \leq p \leq 19$, we determined a_{p^2} , a_{p^3} and a_{p^4} . The first and third number enables one to compute the eigenvalues of Fr_{p^2} . The eigenvalues of Fr_p are the square roots of these numbers. To determine which square root one should take, one can use the value

of a_{p^3} . The first few traces are listed in the following table

p	7	11	13	17	19	23
a_p	0	-116	0	0	-20	0
a_{p^2}	-140	1444	5980	-340	6404	6900
p	29	31	37	41	47	53
a_p	60	24	0	-316	0	0
a_{p^2}	-95116	-82876	-59940	-51516	187060	-471700
p	59	61	67	71	73	79
a_p	-1160	-1116	0	-156	0	-460
a_{p^2}	-146156	-131436	-907180	-814316	27740	-1520396

6. TWO 2-DIMENSIONAL REPRESENTATIONS.

It is important to remark that the traces a_q listed in the previous table are 0 for $q \equiv \pm 2 \pmod{5}$. In this paragraph we will prove this and we will show that this behaviour implies that the Galois representation is induced.

Lemma 6.1. *If $q \equiv \pm 2 \pmod{5}$ then the trace a_q is 0.*

Proof. It follows from proposition 4.2 that $\#\tilde{X}(\mathbb{F}_q) = q^3 + q^2 + q + 1$. The inequality in (5.2) yields:

$$|(k-1)(q^2 + q)| \leq 4q^{3/2}.$$

Hence, $k = 1$ for q big enough. Using the trace formula (5.1) it follows that $a_q = 0$. For small q the a_q are listed in the table (5.3). \square

Let ρ denote the semisimplification of the Galois representation

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Aut}\left(H^3(\tilde{X}_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \otimes \mathbb{Q}_\ell(\sqrt{5})\right).$$

First we will discuss some properties of the characteristic polynomials of the restriction $\rho|_{\mathrm{Gal}(\overline{\mathbb{Q}}/F)}$. If p is inert in the extension F/\mathbb{Q} , and \wp is the prime above p then $\mathrm{Fr}_\wp = (\mathrm{Fr}_p)^2$, and the characteristic polynomial of Fr_\wp is

$$(T^2 - \frac{1}{2}a_{p^2}T + p^6)^2.$$

If p is split, and \wp is a prime above p , then Fr_\wp is a Frobenius element at p , and hence Fr_\wp and Fr_p have the same characteristic polynomial.

Clearly, the characteristic polynomials of $\rho|_{\mathrm{Gal}(\overline{\mathbb{Q}}/F)}$ at inert primes split in two quadratic factors. The polynomials we computed at split primes also split in two quadratic factors over $\mathbb{Q}_\ell(\sqrt{5})$. For example, at $p = 11$ the polynomial is

$$(T^2 + (58 + 2\sqrt{5})T + 1331)(T^2 + (58 - 2\sqrt{5})T + 1331).$$

It turns out that there is no quartic field such that all the characteristic polynomials split further in this quartic field. This suggests the following theorem.

Theorem 6.2. *The restriction $\rho|_{\mathrm{Gal}(\overline{\mathbb{Q}}/F)}$ is reducible. It is a direct sum of two 2-dimensional representations σ and σ' . One has $\rho = \mathrm{Ind}_F^{\mathbb{Q}} \sigma$.*

Proof. Let V denote the vectorspace $H^3(\tilde{X}_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \otimes \overline{\mathbb{Q}_\ell}$, and let $\tilde{\rho} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{Aut}(V)$ be the extension of scalars of ρ . Let χ be the Dirichlet character of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with kernel $\text{Gal}(\overline{\mathbb{Q}}/F)$. By lemma 6.1 we have that $\text{Trace}(\tilde{\rho}(\text{Fr}_p)) = 0$ for all Fr_p not in $\ker \chi$. So $\text{Trace}(\tilde{\rho}(\text{Fr}_p)) = \text{Trace}((\tilde{\rho} \otimes \chi)(\text{Fr}_p))$ for all $p > 5$. It follows that $\tilde{\rho}$ and $\tilde{\rho} \otimes \chi$ are isomorphic. Let $T : V \rightarrow V$ be an isomorphism that intertwines $\tilde{\rho}$ and $\tilde{\rho} \otimes \chi$. So $T(\tilde{\rho}(g)v) = \chi(g)\tilde{\rho}(g)T(v)$ for all $g \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $v \in V$. Let λ be an eigenvalue of T , and V_λ its eigenspace. Clearly, T can not be a scalar map, so V_λ is a proper subspace of V . Since $\tilde{\rho}$ and $\tilde{\rho} \otimes \chi$ actually have coefficients in \mathbb{Q}_ℓ , the eigenvalue λ is contained in an extension of \mathbb{Q}_ℓ of degree at most 4. The image $\tilde{\rho}(\text{Gal}(\overline{\mathbb{Q}}/F))$ commutes with T , so its action on V_λ is a subrepresentation of $\tilde{\rho}|_{\text{Gal}(\overline{\mathbb{Q}}/F)}$. From the remarks made above about the splitting behaviour of the characteristic polynomials it follows that the only possible dimension of V_λ is 2, and that λ is contained in $\mathbb{Q}_\ell(\sqrt{5})$. It also follows T has only one other eigenspace $V_{\lambda'}$, with eigenvalue λ' . Denote the representation on V_λ by σ and the representation on $V_{\lambda'}$ by σ' .

Let $g \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be an element that represents the non-identity element of $\text{Gal}(F/\mathbb{Q})$. Suppose $v \in V_\lambda$. Then

$$T(\tilde{\rho}(g)v) = -\tilde{\rho}(g)(T(v)) = -\lambda\tilde{\rho}(g)v.$$

So $\tilde{\rho}(g)v$ is an eigenvector with eigenvalue $-\lambda = \lambda'$, and $\tilde{\rho}(g)$ sends V_λ to $V_{\lambda'}$. It follows that $\rho = \text{Ind}_F^{\mathbb{Q}} \sigma$. \square

We end this paragraph with a discussion on what could be a different reason for the fact that ρ is induced.

Remember that the variety \tilde{X} was obtained as desingularisation of the closure of an affine variety X . And X is a quotient of a variety Y defined by the equations

$$\begin{aligned} y_1^5 + y_2^5 + y_3^5 + y_1^{-5} + y_2^{-5} + y_3^{-5} &= y_4^5 + y_5^5 + y_6^5 + y_4^{-5} + y_5^{-5} + y_6^{-5}, \\ y_1 y_2 y_3 &= 1, \quad y_4 y_5 y_6 = 1. \end{aligned}$$

We quotient by the action of the group $\mathfrak{S}_3 \times \mathfrak{S}_3$, where the first \mathfrak{S}_3 permutes the coordinates y_1, y_2 and y_3 , and the second permutes y_4, y_5 and y_6 .

On Y there are a lot of automorphisms defined over $\mathbb{Q}(\zeta_5)$. For example, one can send (y_1, y_2, y_3) to $(\zeta_5 y_1, \zeta_5^{-1} y_2, y_3)$, and do something similar with (y_4, y_5, y_6) . One can push down these automorphisms to obtain correspondences on X . It can easily be shown that pushing down an automorphism or its $\mathbb{Q}(\zeta_5)/\mathbb{Q}(\zeta_5 + \zeta_5^{-1})$ -conjugate results in the same correspondence on X . So the correspondences one gets this way are defined over $\mathbb{Q}(\zeta_5 + \zeta_5^{-1}) = F$. They give rise to linear endomorphisms on the cohomology that commute with the $\text{Gal}(\overline{\mathbb{Q}}/F)$ -action. So the eigenspaces of these endomorphisms are subrepresentations of $\tilde{\rho}|_{\text{Gal}(\overline{\mathbb{Q}}/F)}$. We believe that they are 2-dimensional, and that they are infact the representations σ and σ' .

7. THE L -FUNCTION.

Conjecturally, the L -function has an analytic continuation to \mathbb{C} , and satisfies a functional equation (cf. [21]). In this section we explain how one may use this to find candidates for the local L -factors at the bad primes.

Let N be the conductor of ρ , as defined in [21]. It is not known whether N is independent of ℓ . The complete L -function (cf. [21]) has the following shape

$$\Lambda(s) = N^{s/2} (s\pi)^{-2s} \Gamma(s) \Gamma(s-1) L(s).$$

The conjectured functional equation of Λ is

$$(7.1) \quad \Lambda(s) = w \Lambda(4-s)$$

for some $w \in \{\pm 1\}$.

The product $\prod L_p(s)$ is known to converge for $\operatorname{Re}(s) \geq \frac{5}{2}$. In what it follows we use a well known trick that allows one to test numerically the functional equation under the assumption of its analytic continuation (see e.g. [6]). Let $m \geq 0$ be an integer such that all the derivatives $L^{(j)}(2)$ vanish for $0 \leq j \leq m-1$. Choose a real number t . The idea is to integrate the function of one complex variable

$$G(s) := \frac{\Lambda(2+s)t^{-s}}{2\pi i s^{m+1}}$$

along a path in \mathbb{C} around 0. By Cauchy's theorem this integral is $\Lambda^{(m)}(2)/(m!)$. By stretching the path more and more in such a way that in the limit it becomes a union of two lines, (the first going from $r-i\infty$ to $r+i\infty$ and the other going from $-r+i\infty$ to $-r-i\infty$, for some real $r > \frac{1}{2}$), one obtains

$$(7.2) \quad \frac{\Lambda^{(m)}(2)}{m!} = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} \frac{\Lambda(2+s)t^{-s}}{s^{m+1}} ds + \frac{1}{2\pi i} \int_{-r+i\infty}^{-r-i\infty} \frac{\Lambda(2+s)t^{-s}}{s^{m+1}} ds.$$

Define

$$F_m(x) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} \frac{\Gamma(s+2)\Gamma(s)}{x^s s^m} ds.$$

We refer to [6] for a discussion on how to compute values of F_m efficiently.

The first integral of (7.2) is equal to

$$\begin{aligned} & \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} \Gamma(s+2) \frac{\Gamma(s+1)}{s^{m+1}} \left(\frac{4\pi^2}{\sqrt{N}} \right)^{-s-2} t^{-s} \sum_{n=1}^{\infty} \frac{a_n}{n^{s+2}} ds = \\ & \frac{N}{16\pi^4} \sum_{n=1}^{\infty} \frac{a_n}{n^2} \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} \Gamma(s+2)\Gamma(s) \left(\frac{4nt\pi^2}{\sqrt{N}} \right)^{-s} ds = \\ & \frac{N}{16\pi^4} \sum_{n=1}^{\infty} \frac{a_n}{n^2} F_m \left(\frac{4nt\pi^2}{\sqrt{N}} \right). \end{aligned}$$

Using the functional equation (7.1) one can rewrite the second integral of (7.2) as

$$\frac{w}{2\pi i} \int_{r-i\infty}^{r+i\infty} \frac{\Lambda(2+s)t^s}{s^{m+1}} = \frac{wN}{16\pi^4} \sum_{n=1}^{\infty} \frac{a_n}{n^2} F_m \left(\frac{4n\pi^2}{t\sqrt{N}} \right).$$

Consequently, one obtains

$$(7.3) \quad L^{(m)}(2) = m! \sum_{n=1}^{\infty} \frac{a_n}{n^2} F_m \left(\frac{4nt\pi^2}{\sqrt{N}} \right) + wm! \sum_{n=1}^{\infty} \frac{a_n}{n^2} F_m \left(\frac{4n\pi^2}{t\sqrt{N}} \right).$$

We know how to compute $L_p(s)$ for $p > 5$. But we don't know how to compute N , w , and $L_p(s)$ for $p \leq 5$. The idea is that we make a guess for these. For each guess one can use (7.3) to compute $L^{(m)}(2)$ for a few m . We did this computation for several values of t . One knows that $L^{(m)}(2)$ is independent of t . Hence, if the computations give different results for different t , one has probably made wrong guesses. We made new guesses until the computation appeared to be independent of t . The candidates for w are ± 1 . Because the threefold \tilde{X} has bad reduction only at 2, 3 and 5, these are the only primes that can appear in the factorization of N . Hence, one may assume that $N = 2^a 3^b 5^c$ for some integers a , b and c . To guess $L_p(s)$ for $p \leq 5$ we recall that the characteristic polynomial of Frobenius acting on the inertia invariants has degree at most 4. Its roots are conjectured to have absolute value $p^{j/2}$ for some $0 \leq j \leq 3$ (cf. [21]). This limits the possible choices to a finite number. One can reduce this number by using the fact that ρ is induced from a 2-dimensional representation σ . It is well-known that inducing a Galois representation does not change the L -function. Hence, one should find the characteristic polynomials of σ . These polynomials have degree at most 2, and if the representation really ramifies at some p (i.e. $p|N$), their degrees are at most 1. In this case there are only 9 possibilities to be tested, namely the polynomials 1 and $T \pm (N\wp)^j$ for $0 \leq j \leq 3$. The corresponding local L -factors are 1 and $(1 \pm p^{2j-2s})^{-1}$ for $p = 2$ or 3, and 1 and $(1 \pm 5^{j-s})^{-1}$ for $p = 5$.

For $m = 0$, $w = -1$, $N = 2^2 3^2 5^4$ and $L_2(s) = (1 - 2^{2-2s})^{-1}$, $L_3(s) = (1 - 3^{2-2s})^{-1}$ and $L_5(s) = 1$ we computed (7.2) by considering all terms with $n < 30000$ in the infinite sums. We repeated this computation for $t = 1, 1.4, 1.6, 2.0$ and 2.5 . We found the same value 0 upto 48 decimal places. This clearly suggests that the L -series vanishes at $s = 2$. For $m = 1$ we computed (7.2) for $n < 3000$. We took the same values for t , and found that the answer remained constant upto 14 decimal places. The value for $L'(s)$ we found in this way is

$$L'(s) = 2.83811389801282$$

Remark 7.1. *The exponents in the conductor we found are equal to the codimension of the inertia invariants implied by the degree of the local L -factors. This suggests that ρ is only tamely ramified away from ℓ .*

8. THE DESCRIPTION OF A QUATERNION ALGEBRA AND AN EICHLER ORDER.

This paragraph is devoted to the definition of a quaternion algebra and an order. These objects will be used later in the description of the Brandt matrices. The computation of the class and type number of the order are also included. The following notations will hold throughout the rest of the paper.

We denote by F the totally real field $\mathbb{Q}(\sqrt{5})$ and by $w = \frac{1+\sqrt{5}}{2}$ a fundamental unit. We write \mathfrak{o}_F for the ring of integers of F . For $p \in \mathbb{Z}$ a rational prime ideal, we write \wp_p for a prime ideal of \mathfrak{o}_F above p .

Let B be the unique (up to F -linear isomorphism) definite quaternion algebra with center F , ramified at both the infinite places and at \wp_2 and \wp_3 (i.e. with discriminant $D(B/F) = \wp_2 \wp_3$).

Lemma 8.1. *The quaternion algebra B is uniquely determined by the following setting*

$$B = L + LY, \quad L = F(X)$$

where

$$X^2 = -6, \quad Y^2 = \sqrt{5} \frac{1 - \sqrt{5}}{2}$$

for $X, Y \in B$ and $Y \cdot l = \bar{l} \cdot Y$, $\forall l \in L$. Here, $l \mapsto \bar{l}$ denotes the non trivial F -automorphism of L .

In characteristic different from two, B can be shortly described as

$$B = (-6, \sqrt{5} \frac{1 - \sqrt{5}}{2}) = (-6, w - 3).$$

B is the central, simple, 4-dimensional division algebra over F generated by the elements $1, X, Y, XY \in B$ satisfying the following relations

$$X^2 = -6, \quad Y^2 = \sqrt{5} \frac{1 - \sqrt{5}}{2}, \quad XY = -YX.$$

Proof. We only show that B ramifies at \wp_2, \wp_3 and at both the infinite places. For more details on the properties of B , as well as for the proof of its uniqueness up to isomorphism, we refer to [26]. Notice that B is totally definite (*i.e.* $B \otimes_{\mathbb{Q}} \mathbb{R}$ is a division quaternion algebra isomorphic to two copies of the Hamilton's quaternions). In fact, -6 and $\sqrt{5} \frac{1 - \sqrt{5}}{2}$ are totally negative numbers. It remains to show that \wp_2, \wp_3 are the two finite primes where B ramifies. In the local field F_{\wp_3} , the Hilbert symbol (= Hasse invariant) is given by

$$(-6, \sqrt{5} \frac{1 - \sqrt{5}}{2})_{\wp_3} = (\frac{\sqrt{5} \frac{1 - \sqrt{5}}{2}}{3})_{Lg} = -1,$$

where the subscript Lg denotes the Legendre symbol. In fact, $\sqrt{5} \frac{1 - \sqrt{5}}{2}$ is not a square mod \wp_3 . Hence $B_{\wp_3} = B \otimes_F F_{\wp_3}$ is a field. In other words, at the inert prime ideal $\wp_3 \in F$, the algebra B ramifies. At the ramified prime \wp_5 (ramified for the extension F/\mathbb{Q}), the Hilbert symbol is

$$(-6, \sqrt{5} \frac{1 - \sqrt{5}}{2})_{\wp_5} = 1.$$

Hence, the place \wp_5 does not ramify in B . It follows from the definition of the algebra that the only possible place where B could possibly ramify is \wp_2 . This is in fact the case as a quaternion algebra is supposed to ramify at an even number of primes in a field. \square

It follows from Lemma 8.1 that one can define on B an involutive anti-automorphism (*i.e.* the conjugation) as

$$\overline{l_1 + l_2 Y} = \bar{l}_1 - \bar{l}_2 Y, \quad l_1, l_2 \in L.$$

The conjugation appears in the definitions of the reduced trace $Tr_{B/F}$ and the reduced norm $Nr_{B/F}$ on B

$$\begin{aligned} Tr_{B/F}, Nr_{B/F} : B &\rightarrow F \\ Tr_{B/F}(b) &= b + \bar{b}, \quad Nr_{B/F}(b) = b\bar{b}, \quad b \in B. \end{aligned}$$

Note that because B is positive definite, the norm of any element in it is totally positive and every \mathfrak{o}_F -ideal in B has a basis of four elements.

The following order will be used later on in this paragraph. Let

$$(8.1) \quad \mathcal{O}' = \mathfrak{o}_F[1, X, \frac{w}{2} + \frac{1}{2}Y, \frac{w}{2}X + \frac{1}{2}XY].$$

It is easy to verify that \mathcal{O}' is an order of B . We recall that the reduced discriminant $d_r(\Lambda)$ of an order Λ in B , whose associated lattice has a Λ_F -basis $\{e_1, \dots, e_4\}$, is the integral ideal of \mathfrak{o}_F generated by $\sqrt{\det(Tr_{B/F}(e_i \cdot \bar{e}_j))}$. Easy computations show that the reduced discriminant of the order \mathcal{O}' is

$$d_r(\mathcal{O}') = \wp_2 \cdot \wp_3 \cdot \wp_5$$

i.e. \mathcal{O}' has level \wp_5 . Because d_r is square free, it follows that \mathcal{O}' is hereditary (*cf.* [2], Proposition (1.2), p. 304).

Now, we construct a maximal \mathfrak{o}_F -sublattice of the \mathfrak{o}_F -lattice associated to \mathcal{O}' . Using this lattice we plan to define a maximal Eichler suborder of \mathcal{O}' of level 5 (*i.e.* with reduced discriminant $d_r = (\wp_2 \cdot \wp_3) \cdot 5$). Consider the following \mathfrak{o}_F -invertible matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & w \end{pmatrix}.$$

By letting M act, by right multiplication, on the following 4×4 matrix that represents (column way) the \mathfrak{o}_F -lattice associated to \mathcal{O}'

$$\begin{pmatrix} 1 & 0 & w/2 & 0 \\ 0 & 1 & 0 & w/2 \\ 0 & 0 & 1/2 & 1 \\ 0 & 0 & 1 & 1/2 \end{pmatrix},$$

we obtain the following suborder $\mathcal{O} \subset \mathcal{O}'$

$$(8.2) \quad \mathcal{O} = \mathfrak{o}_F[1, X, -\frac{1}{2} + \frac{3-w}{2}Y, -\frac{w}{2} - \frac{w+1}{2}X + \frac{1}{2}Y + \frac{w}{2}XY].$$

It follows from the construction that the index $[\mathfrak{o} : \mathfrak{o}'] = \sqrt{5}$. It is easy to verify that $d_r(\mathcal{O}) = (\wp_2 \cdot \wp_3) \cdot 5$.

Because B ramifies at \wp_2 and \wp_3 , \mathcal{O} is everywhere maximal except at the place \wp_5 . There, the level of \mathcal{O} is two steps below the maximal. Finally, as a consequence of the fact that \mathcal{O}'_{\wp_5} is not a maximal order in a ramified skew-field (*i.e.* B splits at \wp_5), we

can conclude that \mathcal{O}' is an Eichler order. The order is isomorphic, locally at \wp_5 , to the order of the matrices

$$\begin{pmatrix} \mathfrak{o} & \mathfrak{o} \\ \wp_5 \mathfrak{o} & \mathfrak{o} \end{pmatrix},$$

where \mathfrak{o} is the ring of integers of the local field F_{\wp_5} .

It is well known that to an order Λ of B one can (locally) associate an integer called the Eichler invariant. Let v be a finite place of F and let \mathfrak{o}_v be the completion at v of the local ring $\mathfrak{o}_{F,v}$, with maximal ideal \mathfrak{m} and residue field $F_v = \mathfrak{o}_v/\mathfrak{m}$. Then, the Eichler invariant $e(\mathcal{O}_v)$ of $\mathcal{O}_v := B_v \otimes \mathcal{O}$ ($B_v = B \otimes F_v$) is an integer whose value depends upon the description of the quotient \mathcal{O}_v/J_v , where J_v is the Jacobson radical of the ring \mathcal{O}_v . We recall from the theory of Eichler orders developed in [2], that

$$e(\mathcal{O}_v) = \begin{cases} -1 & \text{if } \mathcal{O}_v/J_v \text{ is a quadratic field extension of } (\mathfrak{o}_v/\mathfrak{m}), \\ 0 & \text{if } \mathcal{O}_v/J_v \simeq \mathfrak{o}_v/\mathfrak{m}, \\ 1 & \text{if } \mathcal{O}_v/J_v \simeq \mathfrak{o}_v/\mathfrak{m} \times \mathfrak{o}_v/\mathfrak{m}. \end{cases}$$

If $e(\mathcal{O}_v) = 1$, then \mathcal{O}_v has exactly two minimal overorders and it is an intersection of two uniquely determined maximal orders. Such an order is often called Eichler order and it is isomorphic to the order consisting of matrices

$$\begin{pmatrix} \mathfrak{o}_v & \mathfrak{o}_v \\ \pi^d \mathfrak{o}_v & \mathfrak{o}_v \end{pmatrix}$$

where π denotes a generator (uniformizer) of the maximal ideal \mathfrak{m} and d is a non negative integer. One can fix π so that $v(\pi) = 1$, where v denotes the valuation of F_v that corresponds to \mathfrak{o}_v .

From what we said, it follows that $e(\mathcal{O}'_{\wp_5}) = 1$. Hence, the value of $e(\mathcal{O}_{\wp_5})$ can be either 1 or 0. For, if by absurd $e(\mathcal{O}_{\wp_5}) = -1$, then \mathcal{O}'_{\wp_5} would necessarily have the same Eichler invariant (*cf.* [2] Prop. 3.1), in contradiction with the fact that \mathcal{O}' is an Eichler order.

At the two places $v = \wp_2, \wp_3$, $e(\mathcal{O}_v) = -1$, as \mathcal{O}_v is maximal there. Therefore, the order \mathcal{O} has level 5. Note that the parentheses in $d_r(\mathcal{O}) = (\wp_2 \cdot \wp_3) \cdot 5$ separate the product of the ramified places for B , *i.e.* $D(B/F)$ from the level of the order.

We continue this digression on invariants (locally) associated to orders in quaternion algebras by recalling the definition of the mass of an order. The mass plays a central role in the determination of the class and the type number of an (Eichler) order.

To an order Λ in a quaternion algebra B one associates a rational value $m(\Lambda)$ called the mass. It is well known that the mass depends only upon the level (not on the particular order chosen). Its value appears in the description of the class number $h(\Lambda)$ of the order as the number of distinct classes of left- Λ -ideals (*cf.* [26]). The computation of the mass depends upon certain invariant values associated to the field extension F/\mathbb{Q} and upon the Eichler symbols of the order Λ at each prime dividing the level. A general

description of the mass can be found in [15] (*cf.* Thm 1)

$$(8.3) \quad m(\Lambda) = \frac{2D_K^{3/2}h_K\zeta_K(2)}{(2\pi)^{2n}}Nr_{B/F}(d_r(\Lambda)) \prod_{p|d_r} \frac{1 - Nr(p)^{-2}}{1 - e(\Lambda_p)Nr(p)^{-1}}.$$

Here K denotes a totally real algebraic number field, D_K is its absolute discriminant, h_K is the class number, n is the degree of the field extension and ζ_K is the Dedekind zeta function of K .

The formula (8.3), applied to our case yields, for $n = 2$ and

$$\zeta_F(\wp_2) = D_F^{-3/2}(-2\pi^2)^n|\zeta_F(-1)|, \quad |\zeta_F(-1)| = 1/30, \quad D_F = 5$$

to

$$\begin{aligned} m(\mathcal{O}) &= \frac{2(5)^{3/2}\zeta_F(-1)(5)^{-3/2}(-2\pi^2)^2}{(2\pi)^4} 2^2 3^2 5^2 \frac{1 - 2^4}{1 + 2^2} \cdot \frac{1 - 3^4}{1 + 3^2} \cdot \frac{1 - 5^2}{1 - e(\mathcal{O}_{\wp_5})5^{-1}} = \\ &= \frac{48}{5 - e(\mathcal{O}_{\wp_5})}. \end{aligned}$$

Notice that in the Eichler case (*i.e.* for $e(\mathcal{O}_{\wp_5}) = 1$), one obtains $m(\mathcal{O}) = 12$.

The description of the class number $h(\Lambda)$ of an order Λ depends upon the value of the mass. For the determination of it, we refer to *op.cit.* (*cf.* Thm. 2). Before to state the formula (in the form that applies to our case), we introduce (following *op.cit.*) few further notations.

Let B be the quaternion algebra described in Lemma 8.1 and let \mathcal{O} be the order defined in (8.2). We denote by $C(\mathfrak{o}_F, 1)$ the finite set consisting of all $\alpha \in \mathfrak{o}_F$, with $\alpha^2 - 4 \notin F_v^2$, where v ranges among the set S of places of F where B is a skew-field (*i.e.* it ramifies). It follows that $C(\mathfrak{o}_F, 1) = \{\frac{1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\}$. For each $\alpha \in C(\mathfrak{o}_F, 1)$, we consider the (separable) quadratic extension $L = F[x]$ of F defined by the quadratic equation

$$x^2 - \alpha x + 1 = 0.$$

For each extension $F[x]$, we define the maximal order $\Omega = \mathfrak{o}_F + \mathfrak{o}_F[x]$. The four orders Ω 's so obtained are all isomorphic to

$$\Omega = \mathfrak{o}_F + \mathfrak{o}_F\left[\frac{1}{2}(w + i\sqrt{2+w})\right].$$

The extensions L defined by each of the above quadratic equations (as α varies in $C(\mathfrak{o}_F, 1)$) are unramified at the places \wp_2, \wp_3 and they are ramified at \wp_5 . Under this setting, the Theorem 2 in *op.cit.* computes the class number $h(\mathcal{O})$, as a trace of the Brandt matrix $B(\mathcal{O}, \mathfrak{o}_F)$. In our case the formula reads as

$$(8.4) \quad h(\mathcal{O}) = m(\mathcal{O}) + \sum_{\alpha, \Omega} \prod_{\substack{v \in S \\ v|d_r(\mathcal{O})}} E(\Omega_v, \mathcal{O}_v) \frac{h(\Omega)}{2[\Omega^* : \mathfrak{o}_F^*]}.$$

Here, $E(\Omega_v, \mathcal{O}_v)$ denotes the (local) embedding numbers (*cf.* [2]): *i.e.* the number of classes of all optimal embeddings $\Omega_v \hookrightarrow \mathcal{O}_v$ at each place $v \in S$ (*cf.* [4] and [26]).

To compute the embedding numbers at the two ramified places \wp_2, \wp_3 , one may apply the theory resumed in Vigneras book (*cf. op.cit.* Théorème 3.1) and due to Hijikata. Because the quadratic extension L is unramified at $v = \wp_2, \wp_3$ and because the orders Ω_v and \mathcal{O}_v are both maximal there, it follows that the number of optimal embeddings modulo the action of the units \mathcal{O}_v^* is $E(\Omega_v, \mathcal{O}_v) = 2$. At the place \wp_5 , where L ramifies, one can apply the theory developed by Brzezinski (*cf.* [4] Corollary 1.6 for the Eichler case or [3] Theorem 3.10) to conclude that for $e(\mathcal{O}_{\wp_5}) = 1$, $E(\Omega_{\wp_5}, \mathcal{O}_{\wp_5}) = 0$. From what we said we conclude that

$$h(\mathcal{O}) = m(\mathcal{O}) = 12 \quad \text{for} \quad e(\mathcal{O}_{\wp_5}) = 1.$$

It remains to show that $e(\mathcal{O}_{\wp_5}) = 1$. To this purpose, we use the well known correspondence that relates lattices on quadratic spaces and orders (*cf.* [3]). Namely, ternary lattices define orders in quaternion algebras. To a quaternion algebra B over the field of quotients F of a Dedekind ring R , one can associate the half-regular quadratic space $(B_0, Nr_{B/F})$, where

$$B_0 := \{b \in B : Tr_{B/F}(b) = 0\}.$$

Every R -order Λ in B defines a R -lattice $\Lambda^* \cap B_0$ on B_0 , where

$$\Lambda^* := \{b \in B : Tr_{B/F}(b \cdot \Lambda) \subset R\}.$$

Λ^* is called the dual of the lattice Λ with respect to the trace form on B_0 .

Let suppose from now that R is a principal ideal ring and let e_i be a (finite) R -basis for the lattice Λ^* . To Λ^* one associates the following ternary quadratic form

$$(8.5) \quad q_{\Lambda^*}(X_1, \dots, X_i, \dots) = \frac{1}{Nr_{B/F}(\Lambda^*)} \left(\sum_i Nr_{B/F}(e_i) X_i^2 + \sum_{i < j} Tr_{B/F}(e_i, \bar{e}_j) X_i X_j \right).$$

Note that q_{Λ^*} is (locally) R -integral and it is well defined as R is a principal ideal ring. Furthermore, q_{Λ^*} is primitive, that is, the R -ideal generated by its coefficients is equal to R . It is well known (*cf. op.cit.*) that if p is a prime ideal in R , then the equality $e(R_p) = 1$ implies that q_{Λ^*} splits in a product of two different linear factors when its coefficients are evaluated in the residue field at p . This is what we are going to check in our case.

Because the order \mathcal{O} defined in (8.2) is a Gorenstein, Bass order (*i.e.* $d_r(\mathcal{O})$ is locally cube-free), it follows that the level $l(\mathcal{O}_{\wp_5}) := Nr_{B/F}(\mathcal{O}_{\wp_5}^*)^{-1}$ is equal to the inverse of the (local) reduced discriminant $d_r(\mathcal{O}_{\wp_5}) = 5$.

We re-consider the basis chosen for the order \mathcal{O} , *i.e.*

$$(8.6) \quad \begin{aligned} f_1 &= 1, & f_2 &= X, \\ f_3 &= w + \frac{w}{2}X + Y + \frac{1}{2}XY, & f_4 &= \frac{w}{2} + \frac{1}{2}(w+1)X + \frac{1}{2}Y + \frac{w}{2}XY. \end{aligned}$$

The dual of this basis, with respect to the reduced trace form (*i.e.* g_i such that $Tr_{B/F}(f_i \cdot \bar{g}_j) = \delta_{ij}$) is

$$g_1 = \frac{1}{2} - \frac{1}{10}(1+3w)Y, \quad g_2 = \frac{1}{12}X - \frac{1}{60}(1+3w)XY,$$

$$g_3 = \frac{1}{5}(1+w)Y - \frac{1}{30}wXY, \quad g_4 = -\frac{1}{15}w(3Y - XY).$$

We represent this basis by means of a 4×4 matrix whose i -th column stands for g_i written down on the basis X, Y, XY and 1 . After few column reductions, this matrix takes the following upper triangular form

$$\begin{pmatrix} \frac{1}{6}(2-w) & 0 & \frac{1}{12}(-3+2w) & 0 \\ 0 & -\frac{1}{5}(1+3w) & \frac{1}{5}(1+w) & \frac{1}{10}(1+3w) \\ 0 & 0 & \frac{1}{60}(1+w) & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

The first three columns represent a basis for the \mathfrak{o}_F -lattice $\mathcal{O}^* \cap B_0$. As elements of the quaternion algebra, they are described, in terms of the chosen quaternion basis, as

$$h_1 = \frac{1}{6}(2-w)X, \quad h_2 = -\frac{1}{5}(1+3w)Y,$$

$$h_3 = \frac{1}{12}(-3+2w)X + \frac{1}{5}(1+w)Y + \frac{1}{60}(1+w)XY.$$

The corresponding ternary quadratic form is

$$q_{\mathcal{O}^* \cap B_0} = \frac{5}{6}(5-3w)X_1^2 + (3+4w)X_2^2 + \frac{5}{6}(4-w)X_3^2 + \frac{5}{6}(-8+5w)X_1X_3 - 2(1+2w)X_2X_3.$$

The evaluation of this form in the residue field at \wp_5 gives

$$\bar{q}_{\mathcal{O}^* \cap B_0} = X_2X_3.$$

This shows that $e(\mathcal{O}_{\wp_5}) = 1$. Hence, we can conclude that \mathcal{O} is an Eichler order.

Finally, using Theorem 3 in [15], we determine the type number $t(\mathcal{O})$ of \mathcal{O} . This quantity counts the number of non equivalent types of Eichler orders with reduced discriminant $(\wp_2 \cdot \wp_3) \cdot 5$ in the chosen quaternion algebra B . The formula for the computation of the type number reads in our case as ($S = \{\wp_2, \wp_3, \wp_5\}$)

$$(8.7) \quad m(\mathcal{O}) = t(\mathcal{O}) \prod_{\substack{v \in S \\ v | d_r(\mathcal{O})}} [\Gamma(\mathcal{O}_v) : F_v^* \mathcal{O}_v^*].$$

Here, $[\Gamma(\mathcal{O}_v) : F_v^* \mathcal{O}_v^*]$ is the cardinality of the central Picard group of \mathcal{O}_v , where v denotes a place in S . It is known (*cf.* [16] Satz 2) that this number is 1 when \mathcal{O} is maximal at the place v . On the other hand, for orders whose Eichler invariant is non zero at a place v , the cardinality of the central Picard group is either 1 or 2, depending upon the value of $\text{ord}_v(d_r(\mathcal{O}_v))$ (*i.e.* being 0 or not). Easy computations show that in our case $t(\mathcal{O}) = 3$.

We end this paragraph by resuming the main results proved it it

Proposition 8.2. *Let B be the quaternion algebra with discriminant $D(B/F) = \wp_2 \cdot \wp_3$ defined in Lemma 8.1 and let \mathcal{O} be the Eichler order of level 5, defined in (8.2). Then*

$$h(\mathcal{O}) = 12, \quad t(\mathcal{O}) = 3.$$

9. HILBERT MODULAR FORMS AND QUATERNION ALGEBRAS.

In this paragraph we review the description of the space of quaternionic cusp forms, namely functions defined on a quaternion algebra. One of the main results in the theory is the Jacquet-Langlands Theorem. Shortly said, this theorem relates the systems of eigenvalues found in the spaces of quaternionic cusp forms to certain systems of eigenvalues attached to the spaces of classical modular forms. In this paper, the term classical modular forms means holomorphic Hilbert cusp forms over $F = \mathbb{Q}(\sqrt{5})$ of weight $\mathbf{k} = (2, 4)$ and level $\mathfrak{N} := \wp_2 \wp_3 \cdot 5$: an integral ideal of \mathfrak{o}_F . Our exposition on this correspondence follows the report given by R. Taylor in [25] and by Hijikata, Pizer and Shemanske in [12]. For self-contained reasons we restrict the description to the case of interest.

First, we introduce some notations to which we refer throughout this paragraph and later on in the paper.

Let F denote the totally real quadratic field $\mathbb{Q}(\sqrt{5})$ and let $I = \{\iota_\tau\}_{\tau=1,2}$ be the set of the embeddings $F \hookrightarrow \mathbb{R}$. We shall let B denote the definite quaternion algebra with center F and discriminant $D(B/F) = \wp_2 \wp_3$ introduced in Lemma 8.1. We write \mathcal{O} for the Eichler order of B of level 5 defined in (8.2).

We will consider the following algebraic groups: $\mathfrak{G} = \mathbf{GL}_2$ and G_B over F : $\mathfrak{G}(F) = \mathbf{GL}_2(F)$ and $G_B(F) = B^\times$. Their reduced norm morphisms are $\nu : \mathfrak{G} \rightarrow \mathbf{G}_m$ and $\nu_B : G_B \rightarrow \mathbf{G}_m$. Here \mathbf{G}_m denotes the multiplicative group in one variable considered as an algebraic group defined over F .

Let $\mathbf{A}_F = \mathbf{A}_f \times \mathbf{A}_\infty$ denote the ring of adeles of F , decomposed into its finite and infinite parts. For an algebraic group \mathbf{G} over F , we write $\mathbf{G}_f = \mathbf{G}(\mathbf{A}_f)$ and $\mathbf{G}_\infty = \mathbf{G}(\mathbf{A}_\infty)$.

For v a finite place of F such that $v \nmid d_r(\mathcal{O})$, we fix isomorphisms $M_2(\mathfrak{o}_{F,v}) \simeq \mathcal{O} \otimes_{\mathfrak{o}_F} \mathfrak{o}_{F,v}$. They induce the isomorphism $(G_B)_f \simeq \mathfrak{G}_f$. We choose a subfield K of \mathbb{C} which is Galois over \mathbb{Q} and that splits B , so that there is an isomorphism $i : \mathcal{O} \otimes_{\mathbb{Z}} \mathfrak{o}_K \rightarrow M_2(\mathfrak{o}_K)^I$.

In what it follows we fix $\mathbf{k} = (2, 4) \in \mathbb{Z}^I$.

First, we describe a model for the space of Hilbert modular forms of weight \mathbf{k} . This is done by illustrating the action of the Hecke algebra on it. Our description is given in terms of automorphic functions on the adèle group of \mathfrak{G} and the corresponding Hecke algebra.

Let us consider functions as $f : \mathfrak{G}(\mathbf{A}_F) \rightarrow \mathbb{C}$. For $u = u_f u_\infty \in \mathfrak{G}_f \times \mathfrak{G}_\infty$, define

$$(f|_{\mathbf{k}})(x) = j(u_\infty, z_0)^{-k} \nu(u_\infty)^{\frac{k}{2}} f(xu^{-1}),$$

where

$$\begin{aligned} a) \quad & z_0 = (\sqrt{-1}, \sqrt{-1}) \in \mathcal{Z}^I, \mathcal{Z} \text{ upper half complex plane} \\ b) \quad & j : \mathfrak{G}_\infty \times \mathcal{Z}^I \rightarrow \mathbb{C}^I, \begin{pmatrix} a_\tau & b_\tau \\ c_\tau & d_\tau \end{pmatrix} \times z_\tau \mapsto (c_\tau z_\tau + d_\tau). \end{aligned}$$

For $U \subset (G_B)_f$ an open compact subgroup, one defines

$$\mathcal{S}_{\mathbf{k}}(U) := \{f : \mathfrak{G}(F) \backslash \mathfrak{G}(\mathbf{A}_F) \rightarrow \mathbb{C} \mid \text{under the following conditions 1., 2., 3.}\}$$

1. $f|_{\mathbf{k}} u = f$ for all $u \in UC_\infty$, where $C_\infty = (\mathbb{R}^\times \cdot SO_2(\mathbb{R}))^I \subset \mathfrak{G}_\infty$,
2. $\forall x \in \mathfrak{G}_f$, $f_x : \mathcal{Z}^I \rightarrow \mathbb{C} : uz_0 \mapsto j(u, z_0)^k \nu(u)^{-\frac{k}{2}} f(xu)$, $u \in \mathfrak{G}_\infty$, is holomorphic,
3. $\int_{\mathbf{A}_F/F} f \left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} x \right) da = 0$, $\forall x \in \mathfrak{G}(\mathbf{A}_F)$ da additive Haar measure on \mathbf{A}_F/F .

It is well known that the space $\mathcal{S}_{\mathbf{k}}(U)$ is a model for the space of holomorphic Hilbert modular cusp forms on F of weight \mathbf{k} . The level of these forms is directly related to a choice of the open set U . We refer to [27] (Definition 6.1) for a definition of the space of Hilbert modular forms. In this model, the action of \mathfrak{G}_f is given by right translation.

We define $U_0 = \prod_{\wp} \mathfrak{G}(\mathfrak{o}_{F,\wp})$, where \wp runs over the set of finite, integral, prime ideals in F . From now on we write \mathfrak{P} for the ideal $\wp_2 \wp_3$ in \mathfrak{o}_F product of the two prime inert ideals above 2 and 3 in \mathbb{Z} ; we set $\mathfrak{N} = \mathfrak{P} \cdot 5$. Consider the following open compact subgroup of $(G_B)_f$

$$U(\mathfrak{P}, 5) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0 \mid c \in \mathfrak{P} \cdot 5, a - 1 \in \mathfrak{N} \right\}.$$

We will denote by $\mathcal{S}_{\mathbf{k}}(\mathfrak{P}, 5) = \mathcal{S}_{\mathbf{k}}(U(\mathfrak{P}, 5))$. The space $\mathcal{S}_{\mathbf{k}}(\mathfrak{P}, 5)$ is in one-to-one correspondence with the \mathbb{C} -vector space of holomorphic Hilbert modular cusp forms on F of weight \mathbf{k} and level \mathfrak{N} .

The description of the Hecke algebra on the space $\mathcal{S}_{\mathbf{k}}(\mathfrak{P}, 5)$ goes as follows.

For a prime ideal \wp of F and for the choice of the open compact $U = U(\mathfrak{P}, 5)$, set

$$T_\wp = \left[U \begin{pmatrix} 1 & 0 \\ 0 & \pi_\wp \end{pmatrix} U \right].$$

T_\wp denotes the Hecke operator at \wp , where $\pi_\wp \in \mathbf{A}_f$ is a uniformizer at \wp . There is a similar definition of the Hecke operator, when \wp is replaced by a fractional ideal \mathfrak{a} of F with $(\mathfrak{a}, \mathfrak{N}) = 1$. Note that, although the uniformizer π_\wp is not uniquely determined by \wp , T_\wp is well defined for the choice we made of the open set U (a similar remark holds for the Hecke operator associated to a fractional ideal \mathfrak{a} of F).

As \wp ranges among all finite prime ideals of F (and among all finite integral ideals of F prime to \mathfrak{N}), the \mathbb{Z} -algebra $\mathbf{T}_{\mathbf{k}}(\mathfrak{N}) \subset \text{End}(\mathcal{S}_{\mathbf{k}}(\mathfrak{P}, 5))$ of all these Hecke operators is known to be diagonalizable on the (unique) $\mathbf{T}_{\mathbf{k}}(\mathfrak{N})$ submodule of $\mathcal{S}_{\mathbf{k}}(\mathfrak{P}, 5)$ generated by the eigenforms of the Hecke algebra (*i.e.* $f \in \mathcal{S}_{\mathbf{k}}(\mathfrak{P}, 5)$, such that $f|_{\mathbf{k}} T = \theta_f(T)f$, $\forall T \in \mathbf{T}_{\mathbf{k}}(\mathfrak{N})$).

The space $\mathcal{S}_{\mathbf{k}}(\mathfrak{P}, 5)$ may be equivalently described via the introduction of automorphic forms defined on $G_B(\mathbf{A}_F)$. We will recall this construction, together with the corresponding algebra of Hecke operators and few of their properties.

We denote by $S_{a,b}(\mathbb{C})$ the right $M_2(\mathbb{C})$ -module $\text{Sym}^a(\mathbb{C}^2)$ endowed with the following $M_2(\mathbb{C})$ -action ($m \in M_2(\mathbb{C})$, $s \in \text{Sym}^a(\mathbb{C}^2)$)

$$s \cdot m := (\det m)^b s \text{Sym}^a(m)$$

For $\mathbf{k} = (2, 4)$, we set

$$(9.1) \quad L_{\mathbf{k}}(\mathbb{C}) = S_{0,1}(\mathbb{C}) \otimes S_{2,0}(\mathbb{C}).$$

Each factor of the tensor product is associated to a real embeddings $\iota_\tau \in I$. The group $M_2(\mathbb{C})$ acts (on the right) on $S_{0,1}(\mathbb{C})$ by means of the determinant and via the second symmetric square on $S_{2,0}(\mathbb{C})$. Hence, the space $L_{\mathbf{k}}(\mathbb{C})$ inherits a $(G_B)_\infty$ action via the injection $i : (G_B)_\infty \hookrightarrow \mathfrak{G}(\mathbb{C})^I$ that we have fixed at the beginning of this paragraph.

Now, we consider functions $f : G_B(\mathbf{A}_F) \rightarrow L_{\mathbf{k}}(\mathbb{C})$. For $\tilde{u} = u_f u_\infty \in G_B(\mathbf{A}_F)$, we define

$$(f|_{\mathbf{k}} \tilde{u})(x) = f(x \tilde{u}^{-1}) \cdot u_\infty.$$

The following open compact subgroup of $(G_B)_f$ “corresponds” to $U(\mathfrak{P}, 5)$

$$\mathcal{U} = \prod_{\substack{\wp \in F \\ \wp < \infty}} (\mathcal{O} \otimes_{\mathfrak{o}_F} \mathfrak{o}_{F,\wp})^\times = \prod_{\substack{\wp \in F \\ \wp < \infty}} \mathcal{O}_\wp^\times = \{\tilde{u} = (u_v) \in \prod_v B_v^\times \mid u_v \in \mathcal{O}_v^\times, \forall v < \infty\}.$$

Note that the local description of the order \mathcal{O} is (cf. section 9)

$$\mathcal{O}_\wp^\times \simeq \begin{cases} \mathfrak{G}(\mathfrak{o}_{F,\wp}) & \text{for } \wp \nmid 5, \\ \left\{ \begin{bmatrix} a & b \\ 5c & d \end{bmatrix} \in \mathfrak{G}(\mathfrak{o}_{F,\wp_5}) \mid a, b, c, d \in \mathfrak{o}_{F,\wp_5} \right\} & \text{at } \wp = \wp_5. \end{cases}$$

We set

$$(9.2) \quad S_{\mathbf{k}}^B(\mathcal{U}) := \{f : B^\times \backslash G_B(\mathbf{A}_F) \rightarrow L_{\mathbf{k}}(\mathbb{C}) \mid f|_{\mathbf{k}} \tilde{u} = f, \forall \tilde{u} \in \mathcal{U}(G_B)_\infty\} = \\ = \{f : (G_B)_f / \mathcal{U} \rightarrow L_{\mathbf{k}}(\mathbb{C}), \mid f(b\tilde{x}) = f(\tilde{x}) \cdot b^{-1}, \forall \tilde{x} \in (G_B)_f, b \in B^\times\}.$$

The global units B^\times act via the map $i : B^\times \hookrightarrow \mathfrak{G}(K)^I$ that we have chosen at the beginning of this section. It is worthwhile to remark that the group of the ideles of B can be written as a finite union of distinct double cosets of \mathcal{U} and D^\times ($h(\mathcal{O}) = 12$)

$$(G_B)^\times = \coprod_{\lambda=1}^{12} \mathcal{U} \tilde{g}_\lambda B^\times,$$

where the representatives $\tilde{g}_\lambda = (g_{\lambda_\wp})$ can be (and are) chosen so that $g_{\lambda_\wp} \in \mathcal{O}_\wp^\times$ for all $\wp \mid \mathfrak{N}$. As λ runs between 1 and 12, let $I_\lambda = \mathcal{O} \tilde{g}_\lambda$. The double coset space $\mathcal{X}(\mathcal{U}) = B^\times \backslash (G_B)_f / \mathcal{U}$ can be canonically identified with the right equivalence classes of left \mathcal{O} -ideals I_λ and in particular it is finite. The elements $\tilde{g}_\lambda \in \mathcal{X}(\mathcal{U})$ correspond to the left ideals I_λ of \mathcal{O} , each of which determines a different class in the order. The elements of $S_{\mathbf{k}}^B(\mathcal{U})$ are completely determined by their values at \tilde{g}_λ . In other words, the setting $f \mapsto (f(g_\lambda))$ defines the isomorphism

$$(9.3) \quad S_{\mathbf{k}}^B(\mathcal{U}) \rightarrow \bigoplus_{\lambda=1}^{12} (L_{\mathbf{k}}(\mathbb{C}))^{\mathcal{U} \cap \tilde{g}_\lambda^{-1} B^\times \tilde{g}_\lambda}.$$

In analogy with the previous construction, one may define the Hecke algebra $\mathbf{T}_{\mathbf{k}}^B(\mathcal{U}) \subset \text{End}(S_{\mathbf{k}}^B(\mathcal{U}))$ as the algebra generated by the Hecke operators

$$[\mathcal{U}\tilde{g}\mathcal{U}'] : S_{\mathbf{k}}^B(\mathcal{U}) \rightarrow S_{\mathbf{k}}^B(\mathcal{U}'), \quad f \mapsto \sum f|_{\mathbf{k}}\tilde{g},$$

for \mathcal{U}' open compact subgroup and $\mathcal{U}\tilde{g}\mathcal{U}' = \coprod_i \mathcal{U}g_i\mathcal{U}'$ the decomposition in double cosets. In particular, for $\mathcal{U} = \mathcal{U}'$, we have $[\mathcal{U}\tilde{g}\mathcal{U}] \in \mathbf{T}_{\mathbf{k}}^B(\mathcal{U})$, where $\mathcal{U}\tilde{g}\mathcal{U} = \coprod_i \mathcal{U}g_i$ is the decomposition into disjoint right cosets. The representation ρ of $\mathbf{T}_{\mathbf{k}}^B(\mathcal{U})$ on $S_{\mathbf{k}}^B(\mathcal{U})$ is defined as follows

$$\rho([\mathcal{U}\tilde{g}\mathcal{U}])f(x) = (f|_{\mathbf{k}}\tilde{g})(x) = \sum_i f(g_i x), \quad \tilde{g} = (g_i).$$

Then, one extends ρ to all of $\mathbf{T}_{\mathbf{k}}^B(\mathcal{U})$ by linearity. Because of the identification (9.3), we set $f_{\lambda} = f(\tilde{g}_{\lambda})$, for $f \in S_{\mathbf{k}}^B(\mathcal{U})$. Then, the mapping $\alpha : f \mapsto (f_1, \dots, f_{12})$ describes an isomorphism of $S_{\mathbf{k}}^B(\mathcal{U})$ into $\mathbb{C}^3 \oplus \mathbb{C}^3 \oplus \dots \oplus \mathbb{C}^3$. This sum has exactly $h(\mathcal{O}) = 12$ addenda. By using this identification, one can think of ρ as giving a matrix representation of $\mathbf{T}_{\mathbf{k}}^B(\mathcal{U})$ on $(\mathbb{C}^3)^{12}$. Let $\xi \in \mathfrak{o}_{F, >}$ be a totally positive element in the ring of integers of F . The corresponding Hecke operator is defined as

$$T(\xi) = \sum_{Nr(\tilde{g}) \in \xi \mathcal{U}} [\mathcal{U}\tilde{g}\mathcal{U}], \quad \text{for } \tilde{g} = (g_{\wp}), \quad g_{\wp} \in \mathcal{O}_{\wp}, \quad \forall \wp < \infty.$$

Here Nr denotes the usual extension of the norm $Nr_{B/F}$ to $G_B(\mathbf{A}_F)$. One may represent $\rho(\xi) = \rho(T(\xi))$ as the 12×12 (block) matrix

$$(9.4) \quad B(\xi) = [\rho_{i,j}(\xi)]_{i,j=1,\dots,12}, \quad \rho_{i,j} = \text{pr}_i \cdot \alpha \cdot \rho(\xi) \cdot \alpha^{-1}$$

$$\rho_{i,j} : \mathbb{C}_j^3 \hookrightarrow (\mathbb{C}^3)^{12} \rightarrow S_{\mathbf{k}}^B(\mathfrak{P}, 5) \rightarrow \text{End}(S_{\mathbf{k}}^B(\mathfrak{P}, 5)) \rightarrow (\mathbb{C}^3)^{12} \twoheadrightarrow \mathbb{C}_i^3.$$

The first map on the left denotes the injection of the j -th factor of $(\mathbb{C}^3)^{12}$ into the whole space. The following maps are *resp.* α^{-1} , $\rho(\xi)$, α and finally the i -th projection of $(\mathbb{C}^3)^{12}$ onto \mathbb{C}^3 . In the next paragraph we will give an explicit description of the matrix $B(\xi)$.

We refer to [10] for a proof of the (Hecke equivariant) isomorphism $S_{\mathbf{k}}^B(\mathcal{U}) \simeq S_{\mathbf{k}}(\mathfrak{P}, 5)$. Here, we simply recall the important fact (for which we refer again to *op.cit.*) that these spaces have an equivalent description in terms of representations of *resp.* the groups $G_B(\mathbf{A}_F)$ and $\mathfrak{G}(\mathbf{A}_F)$. To the “new-forms” in the spaces of automorphic forms (*i.e.* eigenfunctions for the Hecke operators) correspond irreducible representations of the groups of the adèles of G_B and \mathfrak{G} .

The second part of this paragraph is expository in nature. We plan to describe how the Jacquet-Langlands construction may help in the process of searching for a particular Hilbert modular form. Roughly said, the idea is to “translate” a local information supplied by the Galois representation ρ (*cf.* paragraph 6) at each place v where ρ ramifies, to an information regarding the properties of the local component π_v of an automorphic representation $\pi = \otimes_v \pi_v$ of $G_B(\mathbf{A}_F)$ and the properties of the underlying quaternion algebra and order. Note that this process makes sense and it is motivated by the knowledge that $\rho \otimes \mathbb{Q}_{\ell}(\sqrt{5}) = \text{Ind}_F^{\mathbb{Q}}(\sigma)$ (*cf.* theorem 6.2).

On the Galois-side, important informations are encoded in the description of the Gamma-factors and in the (Artin) conductor exponents of the functional equation

attached to ρ . These informations determine (uniquely) the weight \mathbf{k} of a modular form over F , the ramification places of a quaternion algebra B and the level of an associated Eichler order \mathcal{O} .

The Jacquet–Langlands Theorem establishes a (Hecke equivariant) injection $\pi \mapsto JL(\pi)$ from the set of (classes of) automorphic representations $\pi = \otimes_v \pi_v$ of $G_B(\mathbf{A}_F) = (B \otimes_F \mathbf{A}_F)^\times$ with $\dim \pi > 1$ (*i.e.* not characters), to the set of cuspidal automorphic representations of $\mathfrak{G}(\mathbf{A}_F)$. The theorem characterizes the image of the “JL” map as the set of cuspidal automorphic representations of $\mathfrak{G}(\mathbf{A}_F)$ which are discrete series (*i.e.* special or supercuspidal at a finite place) at all places at which B ramifies. The representation $JL(\pi)$ is locally characterized, at a place v , only in terms of the related π_v , in the sense that the image only depends on π_v . If v is a finite place at which B splits, then $JL(\pi)_v = \pi_v$.

The description of π_∞ is a consequence of the following remarks. As explained earlier on in the paper, the expected modularity of the two-dimensional representation σ of $\text{Gal}(\bar{F}/F)$ is equivalent to the existence of a holomorphic Hilbert modular form \mathfrak{f} of weight $\mathbf{k} = (2, 4)$. This particular choice for the weight is suggested by the shape of the gamma-factors in the functional equation of the L -function associated to ρ (*cf.* section 7) and by the fact that the functional equations of ρ and σ coincide. A holomorphic Hilbert modular form of weight \mathbf{k} on F verifies the following functional equation (*cf.* [22])

$$L(\mathfrak{f}, s) = L(\mathfrak{f}, 4 - s)$$

where

$$L(\mathfrak{f}, s) := Nr_{F/\mathbb{Q}}(\mathfrak{c}\delta^2)^{s/2}(2\pi)^{-2s}\Gamma(s)\Gamma(s-1)D(\mathfrak{f}, s).$$

We denoted by \mathfrak{c} the conductor of the (Galois) representation associated to \mathfrak{f} (*i.e.* the level N of \mathfrak{f}) and by $\delta = \wp_5 = \sqrt{5}$ the different of the extension F/\mathbb{Q} . The symbol $D(\mathfrak{f}, s)$ means the product of the local, non Archimedean Euler factors. As we explained in details in paragraph 7, our numerical tests determined the conductor of ρ to be $2^2 \cdot 3^2 \cdot 5^4$. Because the scalars extension $\rho \otimes \mathbb{Q}_\ell(\sqrt{5})$ is induced, the level of \mathfrak{f} is forced to be $\mathfrak{c} = (\wp_2 \cdot \wp_3) \cdot 5$. As a consequence of the chosen weight \mathbf{k} , the Archimedean component π_∞ turns out to be isomorphic to the representation

$$\sigma'_\mathbf{k} : (G_B)_\infty \hookrightarrow \mathfrak{G}(\mathbb{C})^I \rightarrow \text{Aut}(L_\mathbf{k}(\mathbb{C})),$$

where the \mathbb{C} -module $L_\mathbf{k}(\mathbb{C})$ was defined in (9.1). By applying the Jacquet–Langlands map, one concludes that $JL(\pi)_\infty \simeq \sigma_\mathbf{k}$ is the discrete series representation of $\mathfrak{G}(\mathbb{R})^I$ whose Langlands’ parameter at each archimedean place is $(\mathbf{k} = (k_\tau) = (2, 4) \in \mathbb{Z}^I)$

$$(9.5) \quad W_\mathbb{R} = \langle \mathbb{C}^\times, j \mid j^2 = -1; jzj^{-1} = \bar{z} \ \forall z \in \mathbb{C}^\times \rangle \rightarrow \mathfrak{G}(\mathbb{C})^I$$

$$z \mapsto \begin{pmatrix} z^{1-k_\tau} & 0 \\ 0 & \bar{z}^{1-k_\tau} \end{pmatrix} (z\bar{z})^{c_\tau} |z|, \quad j \mapsto \begin{pmatrix} 0 & 1 \\ (-1)^{1-k_\tau} & 0 \end{pmatrix}.$$

Here, $(c_\tau) = (1, 0) \in \mathbb{Z}^I$ and $\bar{}$ denotes the complex conjugation.

At the primes $p = 2, 3$ our numerical computations determined the shape of the local Euler factors of the L -function $L(H^3(\bar{X}), s)$ to be of type $(1 - p^{2-2s})^{-1}$ (*i.e.*

trivial central character). This implies that $\pi_v \simeq \det$. Hence, $JL(\pi)_v$ is the special representation which is a subquotient of the principal series representation associated to the pair of characters $|\cdot|^{-1/2}, |\cdot|^{1/2}$. At the ramified place (for the extension F/\mathbb{Q}) \wp_5 (the prime over 5) the local Euler factor was evaluated (numerically) to be 1. In order to test whether $JL(\pi_{\wp_5})$ is a principal or a discrete series (*i.e.* special or supercuspidal), one would need to obtain more information on the L -function. We decided to twist this function with the characters associated to the cyclotomic extension of F with Galois group $(\mathbb{Z}/5\mathbb{Z})^\times$ with the purpose of testing of any change occurring in the local Euler factor. Unfortunately, our numerical computations did not end up with good candidates for the twisted local factors. The problem, however, seems to be not so serious and we are confident that it could be definitively solved with a major accuracy in the setting of the computer program that tests the numerical convergence of the L -series. In fact, twisting the L -function has as consequence that of a possible increase of its conductor. In these cases, more terms in the L -series are needed in order to test their convergency.

10. THE BRANDT MATRICES.

In this paragraph we describe the action of the Hecke operators on the space of forms on a definite quaternion algebra. More precisely, we will define certain square matrices, naturally associated to an Eichler order on a quaternion algebra. We refer to the “basis problem” theory for a proof of the well-known fact that these matrices determine on $S_{\mathbf{k}}^B(\mathcal{U})$ (*cf.* section 9 for notations) the same action as the matrices $B(\xi)$ in (9.4). The theta-series attached to a positive-definite quadratic form define automorphic forms associated to certain open compact subgroups of G_f (*e.g.* $U(\mathfrak{P}, 5)$). Roughly speaking, the “basis problem” asks whether all elements of a certain space of modular forms may be expressed as linear combinations of theta-series attached to the norm form of a definite quaternion algebra (*cf.* [12]). For elliptic modular forms this problem has been studied in great details by Hecke, Eichler, Waldspurger, Hijikata, Pizer and Shemanske. The theory for other modular forms has been started only recently. The works of Shimizu ([23]) and Waldspurger ([30]) give a very useful insight for the solution of the “basis problem” for Hilbert modular forms and for modular forms over \mathbf{GL}_2 . In order to give an explicit construction of the space of automorphic forms on which the “JL” correspondence acts, Shimizu used the spherical functions (*i.e.* functions associated to automorphic forms defined on the group of adeles of G_B) and utilized the theta series defined by A. Weil. He showed that this space consists of cusp forms. Waldspurger used the theory of “newforms” (*i.e.* “classes” of common eigenfunctions for the Hecke operators in the sense of [1]) in the proof of the “basis problem”. His proof can be generalized in order to give a solution of the “basis problem” for Hilbert modular forms (*cf.* [30]). For other type of modular forms (*e.g.* Siegel modular forms) the results are less complete although in progress (*cf.* *e.g.* [30],[2]).

The Brandt matrices describe a representation of the Hecke operators acting on a space of theta series. They satisfy certain commutation relations analogue to those verified by the Hecke operators. Their eigenvalues are subjected to certain constraints expressed by the analogue of the Ramanujan-Petersson conjecture. The aim of this

paragraph is to define these matrices in our context. The explicit realization of them will be given in the next paragraph.

We keep the same notations introduced in the last section. Let start by recalling the following construction. We refer to [19] for more details.

Consider the following reductive group over \mathbb{Q} : $G = \text{Res}_{F/\mathbb{Q}}(B^\times)$. The group $G(\mathbb{R})$ is therefore isomorphic to $(\mathbb{H}^\times)^2$, where \mathbb{H} denote the field of Hamilton's quaternions. The \mathbb{C} -algebra \mathbb{H} can be represented as the subalgebra of $\text{Mat}(2, \mathbb{C})$

$$\mathbb{H} = \left\{ \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \mid z, w \in \mathbb{C} \right\}.$$

In this representation, a basis of \mathbb{H} is

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{I} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

This determines a 2-dimensional (complex) representation

$$\Phi_1 : \mathbb{H}^\times \rightarrow \text{GL}(V).$$

Let choose an isomorphism $V \simeq \mathbb{C}^2$. In terms of the canonical basis $e_1 = (1, 0)$, $e_2 = (0, 1)$ of \mathbb{C}^2 , the matrix representation X_1 of Φ_1 is $(\alpha = x_1 + x_2 \cdot \mathbf{I} + x_3 \cdot \mathbf{J} + x_4 \cdot \mathbf{K} \in \mathbb{H}^\times, z = x_1 + ix_2, w = x_3 + ix_4)$

$$\begin{aligned} X_1(\alpha) &= \begin{pmatrix} x_1 & 0 \\ 0 & x_1 \end{pmatrix} + x_2 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + x_3 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \\ &= \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix} = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}. \end{aligned}$$

The representation Φ_1 induces the determinant representation (character)

$$\det : \mathbb{H}^\times \rightarrow \mathbb{C}^\times.$$

Its description, always in terms of the choice of the canonical basis $\{e_1, e_2\}$ of \mathbb{C}^2 , is

$$X(\alpha) := \det(X_1)(\alpha) = x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

Furthermore, Φ_1 induces a 3-dimensional representation of \mathbb{H}^\times on the 2-th symmetric power $\text{Sym}^2(\mathbb{C}^2) = (\mathbb{C}^2 \otimes \mathbb{C}^2)/K$ (K is the symmetric kernel)

$$\Phi_2 : \mathbb{H}^\times \rightarrow \text{GL}(\mathbb{C}^3).$$

A basis for the space $\text{Sym}^2(\mathbb{C}^2)$ is given by the set of elements $\{e_1^i e_2^{2-i} \mid i = 0, \dots, 2\}$ (the product is considered in the symmetric algebra). The corresponding matrix representation is (the following formula should be read mod K)

$$X_2(\alpha)(e_1^i \otimes e_2^{2-i}) = \bigotimes_1^i (X_1(\alpha)e_1) \otimes \bigotimes_1^{2-i} (X_1(\alpha)e_2).$$

In the previous section (*cf.* (9.1)) we introduced the notation $L_{\mathbf{k}}(\mathbb{C})$ ($\mathbf{k} = (2, 4)$) to indicate the following tensor product of (complex) representations (each factor corresponds to an embedding of $\mathbb{Q}(\sqrt{5})$ in \mathbb{C})

$$L_{\mathbf{k}}(\mathbb{C}) = S_{0,1}(\mathbb{C}) \otimes S_{2,0}(\mathbb{C}).$$

Here, $S_{0,1}(\mathbb{C}) = \mathbb{C}$ denotes the identity representation endowed with a right $M_2(\mathbb{C})$ -module action given by the determinant ($m \in M_2(\mathbb{C})$, $1 \in \text{Id}(\mathbb{C})$)

$$1 \cdot m := \det m.$$

We denote by $S_{2,0}(\mathbb{C})$ the right $M_2(\mathbb{C})$ -module $\text{Sym}^2(\mathbb{C}^2)$. $M_2(\mathbb{C})$ acts on it on the right as ($s \in \text{Sym}^2(\mathbb{C}^2)$)

$$s \cdot m := s \text{Sym}^2(m).$$

In the following, we give an explicitely description of the matrix X_2 attached to the representation Φ_2 .

With the choice of the canonical basis $\{e_1, e_2\}$ for \mathbb{C}^2 , and in terms of the matrix representation $\alpha = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$ ($z = x_1 + ix_2$, $w = x_3 + ix_4$, for $\alpha = x_1 \cdot \mathbf{I} + x_3 \cdot \mathbf{J} + x_4 \cdot \mathbf{K} \in \mathbb{H}$), $\Phi_2(\alpha)(e_2^2)$ is described by

$$X_2(\alpha)(e_2 \otimes e_2) = (X_1(\alpha)e_2) \otimes (X_1(\alpha)e_2) = (-\bar{w}e_1 + \bar{z}e_2)^2 = (-x_3 + ix_4, x_1 - ix_2)^2.$$

Similarly, $\Phi_2(\alpha)(e_1e_2)$ is given by

$$\begin{aligned} X_2(\alpha)(e_1 \otimes e_2) &= (X_1(\alpha)e_1) \otimes (X_1(\alpha)e_2) = (ze_1 + we_2) \cdot (-\bar{w}e_1 + \bar{z}e_2) = \\ &= (x_1 + ix_2, x_3 + ix_4) \cdot (-x_3 + ix_4, x_1 - ix_2), \end{aligned}$$

and $\Phi_2(\alpha)(e_1^2)$ takes the form

$$X_2(\alpha)(e_1 \otimes e_1) = (X_1(\alpha)e_1) \otimes (X_1(\alpha)e_1) = (ze_1 + we_2)^2 = (x_1 + ix_2, x_3 + ix_4)^2.$$

By patching together (row way) these three vectors, we get the description of the (3×3) matrix $X_2(\alpha) = X_2(x_1 + x_2 \cdot \mathbf{I} + x_3 \cdot \mathbf{J} + x_4 \cdot \mathbf{K})$ as

$$(10.1) \quad (p_{ij}(X))_{i,j=1,\dots,3}, \quad \text{for } X = (x_1, \dots, x_4).$$

The entries $p_{ij}(X)$ are the following quadratic, harmonic polynomials with complex coefficients

$$\begin{aligned} p_{11}(X) &= x_1^2 - x_2^2 + 2ix_1x_2, & p_{12}(X) &= 2(x_1x_3 - x_2x_4 + i(x_1x_4 + x_2x_3)), \\ p_{21}(X) &= -x_1x_3 - x_2x_4 + i(x_1x_4 - x_2x_3), & p_{22}(X) &= x_1^2 + x_2^2 - x_3^2 - x_4^2, \\ p_{31}(X) &= x_3^2 - x_4^2 - 2ix_3x_4, & p_{32}(X) &= 2(-x_1x_3 + x_2x_4 + i(x_1x_4 + x_2x_3)), \\ p_{13}(X) &= x_3^2 - x_4^2 + 2ix_3x_4, & p_{23}(X) &= x_1x_3 + x_2x_4 + i(x_1x_4 - x_2x_3), \end{aligned}$$

$$p_{33}(X) = x_1^2 - x_2^2 - 2ix_1x_2.$$

Finally, the action of the determinant representation is given by

$$(10.2) \quad X(\alpha) = x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

Now, we are ready to define the Brandt matrices. Let $I_1 = \mathcal{O}, \dots, I_{12}$ be representatives of all the distinct left \mathcal{O} -ideal classes. Let $\mathcal{O}_j := I_j^{-1}I_j$ be the right order of I_j and let e_j denote the number of units in $\mathcal{O}_j = \frac{1}{Nr_{B/F}(I_j)}\bar{I}_jI_j$. In other words, e_j is the

number of times the positive definite quadratic form $Nr_{B/F}(x)$, $x \in \mathcal{O}_j$ represents 1. Then, for any totally positive element $\xi \in \mathfrak{o}_F$ (*i.e.* $\xi \in \mathfrak{o}_F^+$) we set

$$(10.3) \quad (B(\xi))_{ij} := e_j^{-1} \sum_{\substack{\alpha \in I_j^{-1} I_i \\ Nr_S(\alpha) = \xi}} X(\iota_1 \alpha) \cdot [{}^t X_2(\iota_2 \alpha)]$$

where $\{\iota_\tau\}_{\tau=1,2} = I$ denotes the set of the two real embeddings of $F = \mathbb{Q}(\sqrt{5})$ into \mathbb{R} . The symbol $Nr_S(\alpha)$ stands for the scaled norm of $\alpha \in I_j^{-1} I_i$. If J is an ideal of B , we denote by $Nr(J)_+$ a totally positive generator of the (principal) ideal $Nr_{B/F}(J)$. The scaled norm of an element $\beta \in J$ is the totally positive element $Nr_S(\beta) := \frac{Nr_{B/F}(\beta)}{Nr(J)_+}$. The $(B(\xi))_{ij}$'s are $(3 \cdot 12) \times (3 \cdot 12)$ square \mathbb{C} -matrices and they are divided into 144 (3×3) -blocks.

As ξ varies among the set of totally positive elements of \mathfrak{o}_F , the entries of the Brandt matrix series

$$(10.4) \quad \Theta(z) = \sum_{\xi \in \mathfrak{o}_F^+} B(\xi) \exp(2\pi\sqrt{-1} \xi z), \quad z \in \mathfrak{H} = \text{upper half plane}$$

represent holomorphic Hilbert cusp forms of weight \mathbf{k} on $S_{\mathbf{k}}(\mathfrak{P}, 5)$ on which the Brandt matrices act. In particular, these matrices satisfy identities similar to those fulfilled by the Hecke operators.

It is worth to recall the following elementary property of the totally positive elements of F . For a proof of it, we refer to any introductory book in algebraic number theory (w a fundamental unit in F)

$$\mathfrak{o}_F^+ = \{\xi = a + bw \mid a, b \in \mathbb{Z}, a > 0, \text{ and } -\frac{a}{w} < b < \frac{a}{w-1}\}.$$

Using this property, we can order the elements of \mathfrak{o}_F^+ lexicographically (*i.e.* first by a and then by b). Hence, the sum in (10.4) is well defined.

It follows from the theory of the “basis problem” (*cf.* [12]) that $(B(\xi))_{ij}$ give a representation of the Hecke operators on the space of cusp forms on the quaternion algebra B . Namely, they describe the action of $(G_B)_\infty$ on $L_{\mathbf{k}}(\mathbb{C})$ via the chosen map $i : (G_B)_\infty \hookrightarrow \mathfrak{G}(\mathbb{C})^I$ as we explained in the last paragraph. The explicit expression of these matrices is subordinated to the knowledge of the left ideal classes I_j , the right orders \mathcal{O}_j and the associated θ -series. This part will be described in the next section.

11. THE DESCRIPTION OF THE ALGORITHM.

Let $F = \mathbb{Q}(\sqrt{5})$ and let $w = \frac{1+\sqrt{5}}{2}$ be a fundamental unit. Let B be the quaternion algebra with center F as defined in Lemma 8.1. We recall the definition of the Eichler order \mathcal{O} of level 5 defined in section 8

$$\mathcal{O} = \mathfrak{o}_F[1, X, -\frac{1}{2} + \frac{3-w}{2}Y, -\frac{w}{2} - \frac{w+1}{2}X + \frac{1}{2}Y + \frac{w}{2}XY].$$

The order \mathcal{O} contains 12 different left-ideals classes whereas the algebra B has 3 distinct isomorphism classes of orders with the same level as \mathcal{O} .

The method used for finding the left-ideal classes is similar to that applied for computing the ideal class group of a number field. The properties $h(F) = 1$ and

$Nr_{F/\mathbb{Q}}(w) = -1$ guarantee that any ideal of \mathfrak{o}_F can be generated by a totally positive element and that every ideal of B has a basis over \mathfrak{o}_F .

For a fixed $\alpha \in B \setminus F$, the quadratic field extension $K = F(\alpha)$ is contained in B . The non-trivial field automorphism σ of K that fixes F may be thought of as the conjugation in B : $\alpha^\sigma = \bar{\alpha}$. In this way, the relative norm $Nr_{K/F}$ becomes the restriction $Nr|_K$ to K of the norm $Nr_{B/F}$ defined on B .

If \mathcal{I} is an ideal of \mathfrak{o}_K (the ring of integers of K), the modulus $I = \mathcal{O}\mathcal{I}$ represents a left-ideal of \mathcal{O} . This is because $\mathcal{O}(\mathcal{O}\mathcal{I}) = \mathcal{O}\mathcal{I}$. Furthermore: $Nr_{B/F}(I) = Nr|_K(\mathcal{I})$, as $1 \in \mathcal{O}$. Finally, the representatives of the same class in the ideal class group of K give rise to elements that belong to the same class as left ideals in \mathcal{O} .

To an ideal in the quaternion algebra B , we associate its θ -series. This series catalogues, in a way that we will make clear quite soon, the number of elements in the ideal with a fixed norm. More precisely, if I is a \mathfrak{o}_F -ideal of B , one defines the θ -series of I as

$$(11.1) \quad \theta_I(\tau) = \sum_{\alpha \in I} \exp(\tau Nr_S(\alpha)) = \sum_{\xi \in \mathfrak{o}_F^+} c_{\xi, I} \exp(\tau \xi), \text{ where} \\ c_{\xi, I} = \#\{\alpha \in I \mid Nr_S(\alpha) = \xi\}.$$

For the notations used here, we refer to (10.3). The symbols $c_{\xi, I}$ are the representation numbers of ξ in I . Note that $c_{\xi, I}$ is finite for every ξ and I . This is an easy consequence of the fact that B is totally definite. For a fixed choice of ξ , the definition of the representation numbers is independent of the choice of the positive generator of $Nr_{B/F}(I)$ (i.e. $Nr_+(I)$) involved in the definition of the scaled norm Nr_S (i.e. any two choices for $Nr_+(I)$ differ by a totally positive unit in F). Thus, the θ -series are well defined objects. These series are used to provide a necessary condition for testing whether two ideals in B belong to the same class. The proof of the following proposition is immediate.

Proposition 11.1. *If I and J are \mathfrak{o}_F -ideals of B ,*

$$J = \gamma_1 I \gamma_2, \quad \gamma_i \in B^\times \quad \text{then} \quad c_{\xi, J} = c_{\xi, I} \quad \forall \xi \in \mathfrak{o}_F^+.$$

Notice that two ideals (or orders) in B may have the same θ -series and generate different classes. In order to check whether two left \mathcal{O} -ideals belong to different classes, one is sometimes required to use the following necessary and sufficient condition.

Proposition 11.2. *Let I and J be two left \mathcal{O} -ideals for an Eichler order \mathcal{O} . Then, I and J belong to the same ideal class if and only if there exists an element $\alpha \in \bar{J}I$ such that $Nr_{B/F}(\alpha) = Nr_{B/F}(I)Nr_{B/F}(J)$.*

Proof. We refer to [19] Proposition 1.8, for the proof. The computations written down in *op.cit.* generalize easily to any quaternion algebra over a number field. \square

The criterium of Proposition 11.2 is the most convenient for testing ideals and orders in B . It is clear though that its accuracy is paid by time consuming!

The following, is a concise description of the algorithm that we used in the process of testing ideal classes. This procedure requires to find the representation numbers c_ξ

in the most efficient way possible. The c_ξ 's describe the number of times the quadratic form Nr_S represents the totally positive definite element $\xi \in \mathfrak{o}_F$.

If I is an ideal of B with basis $\{X_1, \dots, X_4\}$, any element $b \in I$ can be written in the form $b = (b_1 + b_5w)X_1 + (b_2 + b_6w)X_2 + (b_3 + b_7w)X_3 + (b_4 + b_8w)X_4$. Its scaled norm is then $Nr_S(b) = Q_1(Y) + Q_2(Y)w$, where $Y = [b_1, \dots, b_8] \in \mathbb{Z}^8$. The functions Q_1 and Q_2 are quadratic forms with coefficients in \mathbb{Q} and values in \mathbb{Z} . Furthermore, Q_1 is (symmetric) positive definite because the quaternion algebra is totally definite. This property holds regardless of the chosen basis for I .

If $\xi = \xi_1 + \xi_2w \in \mathfrak{o}_F$, then the finite set $\{Y \in \mathbb{Z}^8 \mid Q_1(Y) = \xi_1\}$ (as well as the similar set associated to $Q_2(Y)$) is computable through a process of diagonalization of $Q_1(Y)$ obtained by using the “completion of square” method. The efficiency of this algorithm depends upon the choice of the basis for I . It can be easily seen (and quite natural to expect) that the optimal choice for the basis is the one associated to a Hermite normal form of the 4×4 matrix that describes a basis for the ideal. The existence of such a form is a consequence of the fact that F is an Euclidean domain (although the process is still possible in the more general setting of a UFD domain).

Said that, it is immediate to verify that an efficient use of the necessary and sufficient condition of proposition 11.2 relies on the process of first multiplying two 4×4 matrices (previously reduced in Hermite normal form) and then using the Hermite normal form reduction to reduce a 4×16 matrix into a 4×4 upper-triangular.

The main part of the process is the search of the complete set of solutions for the system: $Q_1(Y) = \xi_1M_1$, $Q_2(Y) = \xi_2M_2$, in the way explained before and for $Nr_{B/F}(I)Nr_{B/F}(J) = M_1 + M_2w$. This is the most lengthy and tedious part. Its efficiency depends on a well-written implementable computer algorithm for finding the solutions of a given quadratic form previously written as a sum of squares with positive integer coefficients.

Now, we pass to the description of the left-ideals and their right orders.

We found the twelve left-ideals of B by considering five distinct quadratic extensions $K = F(\alpha)$, $\alpha \in B$. We do not claim that the number and the choice of the extensions was the best possible. It is quite plausible that one may find the whole bunch of classes by considering a single extension K as above. In the table below we have described these extensions together with their minimal polynomials p_i over F and the class numbers of the quartic extensions K_i/\mathbb{Q} . We denoted by f_i $i = 1, \dots, 4$ the basis of the order \mathcal{O} as defined in (8.6).

K_i	$\alpha_i \in \mathcal{O}$	$p_i(x)$	$h(K_i)$
$F(\alpha_1)$	f_2	$x^2 + 6$	4
$F(\alpha_2)$	$f_2 + 3f_3 - f_4$	$x^2 - 5wx + 85 - 18w$	22
$F(\alpha_3)$	$(2 - w)f_1 - f_2 - (w + 1)f_3 + wf_4$	$x^2 - (3 - 5w)x + 17$	5
$F(\alpha_4)$	$-f_2 - (w + 2)f_3 + (2w - 1)f_4$	$x^2 + 5wx + 31$	9
$F(\alpha_5)$	$-(5w + 2)f_1 + 2(2w + 1)f_3 + (1 - 4w)f_4$	$x^2 + wx + 47$	17

We write

$$\begin{aligned}\alpha_1 &= X, \quad \alpha_2 = \frac{5}{2}w + \left(\frac{1}{2} + w\right)X + \frac{5}{2}Y + \frac{1}{2}(3 - w)XY, \\ \alpha_3 &= \frac{1}{2}(3 - 5w) - X - \frac{1}{2}(2 + w)Y, \quad \alpha_4 = -\frac{5}{2}w - X - \frac{5}{2}Y, \\ \alpha_5 &= -\frac{w}{2} + \frac{1}{2}(1 - w)X + \left(\frac{5}{2} + 2w\right)Y + \frac{1}{2}(-2 + w)XY.\end{aligned}$$

In the following table we have collected together the definitions of the ideals $\mathcal{I}_i = (a_i + wb_i, \gamma_i)$ ($i = 1, \dots, 12$) in the quartic extensions K_i . We used the necessary and sufficient condition of proposition 11.2 to verify that the associated left- \mathcal{O} -ideals $\mathcal{OI}_i = I_i$ define twelve distinct classes.

\mathcal{I}_i	K_i	$a_i + wb_i$	γ_i	$I_i p \in \mathbb{Z}$
\mathcal{I}_1	F	1		
\mathcal{I}_2	$F(\alpha_1)$	$2 + w$	$2 + \alpha_1$	5
\mathcal{I}_3	$F(\alpha_1)$	2	$2w + \alpha_1$	2
\mathcal{I}_4	$F(\alpha_1)$	$2 + w$	$3 + \alpha_1$	5
\mathcal{I}_5	$F(\alpha_2)$	$2 + w$	$-2 + \alpha_2$	5
\mathcal{I}_6	$F(\alpha_2)$	$7 - 2w$	$16 + \alpha_2$	31
\mathcal{I}_7	$F(\alpha_2)$	$9 + 13w$	$10 + \alpha_2$	29
\mathcal{I}_8	$F(\alpha_3)$	$4 - w$	$8 + \alpha_3$	11
\mathcal{I}_9	$F(\alpha_3)$	$4 - w$	$9 + \alpha_3$	11
\mathcal{I}_{10}	$F(\alpha_4)$	7	$6 + 4w + \alpha_4$	7
\mathcal{I}_{11}	$F(\alpha_5)$	$3w + 5$	$13 + \alpha_5$	31
\mathcal{I}_{12}	$F(\alpha_5)$	17	$12(1 + w) + \alpha_5$	17

The initial coefficients of the theta series of the ideals I_i are tabulated below. Notice that since \mathcal{O} has 2 elements of norm 1, every θ -series coefficient is a multiple of 2.

$$\zeta = a + bw$$

Ideals	1	2	$2 + w$	$3 - w$	3	$3 + w$	$4 - w$	4	$4 + w$	$4 + 2w$	5	10
$I_1 = \mathcal{O}$	2	0	0	0	0	4	4	2	0	0	10	20
I_2	0	0	10	10	0	2	2	0	4	2	18	0
I_3	0	2	0	0	2	0	0	0	0	0	20	10
I_4	0	0	2	2	0	2	2	0	4	10	0	18
I_5	0	0	2	2	0	2	2	0	4	0	10	18
I_6	0	0	0	0	0	2	2	0	4	2	18	10
I_7	0	0	0	0	0	2	2	0	4	2	18	10
I_8	0	0	0	0	0	2	2	0	4	2	18	10
I_9	0	0	2	2	0	2	2	0	4	0	10	18
I_{10}	0	0	2	2	0	2	2	0	4	0	10	18
I_{11}	0	0	0	0	0	2	2	0	4	2	18	10
I_{12}	0	0	2	2	0	2	2	0	4	0	10	18

The following table shows few of the coefficients of the θ -series of the right orders $\mathcal{O}_i := I_i^{-1}I_i$.

$$\zeta = a + bw$$

Right-Orders	e_i	2	3	4	5	6	7	8	9	10	11	$11 + w$	$12 - w$	$13 + w$
$\mathcal{O}_1 = \mathcal{O}$	2	0	0	2	10	2	20	0	2	20	28	4	4	44
\mathcal{O}_2	2	0	0	2	10	2	20	0	2	20	28	4	4	44
\mathcal{O}_3	2	0	0	2	10	2	20	0	2	20	28	4	4	44
\mathcal{O}_4	2	0	0	2	10	2	20	0	2	20	28	4	4	44
\mathcal{O}_5	2	0	0	2	26	0	8	0	2	0	20	14	16	20
\mathcal{O}_6	2	0	0	2	26	0	8	0	2	0	20	16	14	16
\mathcal{O}_7	2	0	0	2	26	0	8	0	2	0	20	16	14	16
\mathcal{O}_8	2	0	0	2	26	0	8	0	2	0	20	16	14	16
\mathcal{O}_9	2	0	0	2	26	0	8	0	2	0	20	14	16	20
\mathcal{O}_{10}	2	0	0	2	26	0	8	0	2	0	20	16	14	16
\mathcal{O}_{11}	2	0	0	2	26	0	8	0	2	0	20	14	16	20
\mathcal{O}_{12}	2	0	0	2	26	0	8	0	2	0	20	14	16	20

Because $t(\mathcal{O}) = 3$, there are only three different types (*i.e.* conjugacy classes by elements of B^\times) of orders of B with level 5. One can classify them by looking at the corresponding θ -coefficients. The above table shows that at $\xi = 5$ one can distinguish the first equivalence class. At $\xi = 11 + w$ the three classes become all distinct. We have chosen \mathcal{O} , \mathcal{O}_5 and \mathcal{O}_6 as representatives for these classes. The lattices spanned by $\bar{I}_5 I_5$ and by $\bar{I}_6 I_6$ are given by the following upper triangular matrices (each column describes the coefficients of *resp.* 1, X , Y , XY). The ratio in front of each matrix represents the norm of the related \mathcal{O} -left-ideal (notice that $\mathcal{O}_j = \frac{1}{Nr_{B/F}(I_j)} \bar{I}_j I_j$)

$$\mathcal{O}_5 = \mathfrak{o}_F \left[\frac{1}{2+w} \begin{pmatrix} 4w+3 & 0 & -(2w+\frac{3}{2}) & -\frac{1}{2}(11w+7) \\ 0 & 5w & -(3w+1) & w-\frac{1}{2} \\ 0 & 0 & -\frac{5}{2}(w+1) & -(\frac{3}{2}w+1) \\ 0 & 0 & 0 & \frac{1}{2}(w-1) \end{pmatrix} \right],$$

$$\mathcal{O}_6 = \mathfrak{o}_F \left[\frac{1}{7-2w} \begin{pmatrix} 2w-7 & 0 & -w+\frac{7}{2} & 0 \\ 0 & 73w+39 & 30w+19 & -(\frac{55}{2}w+19) \\ 0 & 0 & \frac{1}{2}(11w-23) & 2(w-2) \\ 0 & 0 & 0 & w+\frac{1}{2} \end{pmatrix} \right].$$

For the computation of the Brandt matrices $(B(\xi))_{i,j}$ we needed to know, for the first few totally positive $\xi \in \mathfrak{o}_F$, the number of elements $\alpha \in I_j^{-1}I_i$ with scaled norm ξ . Due to the symmetry of the Brandt matrices (*cf.* [19]), it suffices to compute these numbers for the 78 ideals $I_j^{-1}I_i$ for $j \geq i$.

Because the Brandt matrices generate a commutative semisimple ring it is possible to diagonalize them simultaneously. This means that one can choose a finite set of totally positive elements ξ and for each of them compute the corresponding Brandt matrices and then search for a simultaneous diagonalization. In our case we only needed to verify

that among the eigenvectors there was one whose eigenvalues mapped down to \mathbb{Q} via the norm map, match with the traces of the Frobenius listed in paragraph 5.

Note also that it is enough to compute the characteristic polynomial (of degree 36) of the matrix ${}^tX_2(\iota_2(\alpha))$ (*cf.* (10.3)) for a particular choice of ξ and then carry on the computations for it. We decided to choose $\xi = 3 + w$ (one of the two primes in F above 11) and factor the related polynomial in F . This polynomial splits in 10 linear factors (each of which appears with multiplicity greater than one) and a factor of degree 3 (counted with multiplicity 2). Note that one has to take into account the presence of the determinant representation in the definition of $(B(\xi))_{ij}$. In fact, it turned out that one of the \mathfrak{o}_F -roots of the associated Brandt polynomial (appearing with multiplicity 3) is $-4(4+w)(4-w) = 4(-15+w)$. The factor $-4(4+w)$ represents the root associated to the representation X_2 , whereas $4-w$ describes the action of the determinant. The first number coincides with the degree-one coefficient of one of the two quadratic factors in which the characteristic polynomial of the Frobenius at 11 splits (*cf.* section 5).

Using the same technique we computed the characteristic polynomial of the matrix $(B(\xi))_{i,j}$ at the inert prime $\xi = 7$. This polynomial has 16 rational roots (counted with multiplicities). One of these is $-10 \cdot 7 = -70$ (with multiplicity 4). Here, -10 is the root associated to X_2 . Again, the square of this number coincides with the degree-two coefficient in the characteristic polynomial of the Frobenius at 7.

At this point, we considered the common eigenspace of the two matrices at the eigenvalue of interest. This is a one-dimensional space. Its eigenvector ε was the natural candidate for representing the Hilbert modular form \mathfrak{f} (*cf.* paragraph 9). To compute further eigenvalues associated to ε , it was enough to work with a particular column of the Brandt matrix and multiply that with ε .

The following is the description of the (transpose of the) eigenvector ε (we denote by $v = \sqrt{3-w}$):

$$\begin{aligned} \varepsilon = [& 0, -w-2, 0, 0, 1, 0, 0, -\frac{1}{2}w-1, 0, 0, 1, 0, (-\frac{3}{25}w + \frac{4}{25})v\sqrt{-6} + (\frac{3}{50}w - \frac{2}{25})v, \frac{1}{10}w - \\ & \frac{3}{10}, (-\frac{3}{25}w + \frac{4}{25})v\sqrt{-6} + (-\frac{3}{50}w + \frac{2}{25})v, (\frac{833}{4805}w + \frac{486}{4805})v\sqrt{-6} + (\frac{467}{9610}w + \frac{142}{4805})v, -\frac{183}{1922}w - \\ & \frac{101}{1922}, (\frac{833}{4805}w + \frac{486}{4805})v\sqrt{-6} + (-\frac{467}{9610}w - \frac{142}{4805})v, (\frac{63}{4205}w - \frac{4}{4205})v\sqrt{-6} + (\frac{847}{8410}w - \frac{521}{8410})v, -\frac{2}{841}w + \\ & \frac{67}{1682}, (\frac{63}{4205}w - \frac{4}{4205})v\sqrt{-6} + (-\frac{847}{8410}w + \frac{521}{8410})v, (\frac{4}{605}w + \frac{23}{605})v\sqrt{-6} + (-\frac{169}{1210}w + \frac{87}{1210})v, -\frac{2}{121}w - \\ & \frac{23}{242}, (\frac{4}{605}w + \frac{23}{605})v\sqrt{-6} + (\frac{169}{1210}w - \frac{87}{1210})v, (\frac{89}{605}w + \frac{58}{605})v\sqrt{-6} + (\frac{441}{1210}w + \frac{267}{1210})v, \frac{60}{121}w + \\ & \frac{85}{242}, (\frac{89}{605}w + \frac{58}{605})v\sqrt{-6} + (-\frac{441}{1210}w - \frac{267}{1210})v, (\frac{4}{245}w + \frac{3}{245})v\sqrt{-6} + (\frac{111}{490}w + \frac{31}{245})v, -\frac{5}{98}w - \\ & \frac{5}{98}, (\frac{4}{245}w + \frac{3}{245})v\sqrt{-6} + (-\frac{111}{490}w - \frac{31}{245})v, (\frac{19}{4805}w + \frac{63}{4805})v\sqrt{-6} + (\frac{1221}{9610}w - \frac{884}{4805})v, \frac{107}{1922}w - \\ & \frac{151}{1922}, (\frac{19}{4805}w + \frac{63}{4805})v\sqrt{-6} + (-\frac{1221}{9610}w + \frac{884}{4805})v, (-\frac{1}{85}w + \frac{16}{1445})v\sqrt{-6} + (-\frac{33}{2890}w - \frac{11}{2890})v, \frac{4}{289}w - \\ & \frac{19}{578}, (-\frac{1}{85}w + \frac{16}{1445})v\sqrt{-6} + (\frac{33}{2890}w + \frac{11}{2890})v]. \end{aligned}$$

The following table resumes the first few eigenvalues of ε when ξ is a prime (one can compare these values with those listed in table (5.3)).

$\xi = a + wb$	$\xi p \in \mathbb{Z}$	ε
1		1
2	2	4
3	3	9
$-1 + 2w$	5	0
7	7	-70
$3 + w$	11	$4(-15 + w)$
$4 - w$	11	$-4(14 + w)$
13	13	2990
17	17	-170
$4 + w$	19	$4(12 - 29w)$
$5 - w$	19	$-4(17 - 29w)$
23	23	3450
$5 + w$	29	$2(19 - 8w)$
$6 - w$	29	$2(11 + 8w)$
$7 - 2w$	31	$24(3 - 5w)$
$5 + 2w$	31	$-24(2 - 5w)$
37	37	-29970
$7 - w$	41	$-2(13 + 132w)$
$6 + w$	41	$-2(145 + 132w)$
43	43	149210
47	47	93530
53	53	-235850
$2w + 7$	59	$-4(141 + 8w)$
$9 - 2w$	59	$4(149 - 8w)$

Theorem 11.3. *For each ξ the eigenvalue of $B(\xi)$ corresponding to the above eigenvector is in F .*

Proof. Suppose $\alpha = x_1 + x_2 \cdot \mathbf{I} + x_3 \cdot \mathbf{J} + x_4 \cdot \mathbf{K}$ is an element of B . Then $x_1 \in F$, $x_2 \in F \cdot \sqrt{6}$, $x_3 \in F \cdot v$ and $x_4 \in F \cdot v\sqrt{6}$. Recall the definition of $X_2(\alpha)$ from (10.1) in terms of harmonic polynomials p_{ij} . From the definition of the p_{ij} it immediately follows that $X_2(\alpha)_{ij} \in F(\sqrt{-6})$ if $(i, j) = (1, 1), (1, 3), (3, 1), (3, 3)$ or $(2, 2)$, and $X_2(\alpha)_{ij} \in F(\sqrt{-6}) \cdot v$ otherwise. From the definition of $B(\xi)$ it follows that $B(\xi)_{ij} \in F(\sqrt{-6})$ if (i, j) is congruent to $(1, 1), (1, 3), (3, 1), (3, 3)$ or $(2, 2) \pmod{3}$, and $B(\xi)_{ij} \in F(\sqrt{-6}) \cdot v$ otherwise. Note that the eigenvector ε satisfies $\varepsilon_i \in F(\sqrt{-6})$ if $i \equiv 2 \pmod{3}$, and $\varepsilon_i \in F(\sqrt{-6}) \cdot v$ otherwise. So $B(\xi)\varepsilon$ also has this property, hence the eigenvalue of ε is in $F(\sqrt{-6})$.

It is known that eigenvalues of the Hecke action on Hilbert modular forms are contained in a totally real field. This implies that the eigenvalue of ε is in F . \square

REFERENCES

- [1] A. O. L. Atkin, J. Lehner *Hecke Operators on $\Gamma_0(m)$* , Math. Ann. **185** (1970), 134–160.
- [2] J. Brzezinski *On Orders in Quaternion Algebras*, Comm. Algebra **11** (1983) No. 5, 501–522.
- [3] ——— *On Automorphisms of Quaternion Orders*, J. Reine Angew. Math. **403** (1990), 166–186.
- [4] ——— *On Embeddings Numbers into Quaternion Orders*, Comment. Math. Helvetici **66** (1991), 302–318.
- [5] S. Böcherer *Siegel Modular Forms and Theta Series*, Proc. Sympos. Pure Math. **49**, Amer. Math. Soc., Providence RI, (1989).
- [6] J. Buhler, C. Schoen, J. Top, *Cycles, L-functions and triple products of elliptic curves*, J. Reine Angew. Math **492** (1997), 93–133.
- [7] S.V. Chmutov *Examples of Projective Surfaces with Many Singularities*, J. Algebraic Geom. **1** No. 2, (1992), 191–196.
- [8] ——— *Extremal Distribution of Critical Points and Critical Values*, Singularity Theory Trieste (1991), World Science Publ., River Edge NJ 1995, 192–205.
- [9] G. Faltings *Endlichkeitssätze für Abelsche Varietäten über Zahlkörpern*, Invent. Math. **73** (1983), 349–366.
- [10] S. Gelbart *Authomorphic Forms on Adeles Groups*, Princeton Univ. Press (1975).
- [11] F. Hirzebruch *Some Examples of Threefolds with Trivial Canonical Bundle*, Collected Papers **II** No. 75, Springer-Verlag, Heidelberg (1987), 757–770.
- [12] H. Hijikata, A. Pizer, T. Shemanske *Orders in Quaternion Algebras*, J. Reine Angew. Math **394** (1989), 59–106.
- [13] M. E. Hoffman, W. D. Withers *Generalized Chebyshev Polynomials Associated with Affine Weyl Groups*, Trans. American Math. Society, **308** No. 1, (1988), 91–104.
- [14] K. Kimura *An Example of Algebraic Cycles with Non Trivial Abel-Jacobi Image*, Journal of Algebra **222** No. 1, (1999), 129–145.
- [15] O. Körner *Traces of Eichler-Brandt Matrices and Type Numbers of Quaternion Orders*, Proc. Indian Acad. Sci. **97** (1987), 189–199.
- [16] ——— *Über die Zentrale Picard-Gruppe und die Einheiten Lokaler Quaternionenordnungen*, Manuscripta Math. **52** (1985), 203–225.
- [17] R. Lidl, G. L. Mullen, G. Turnwald *Dickson Polynomials*, Pitman Monographs and Surveys in Pure and Applied Mathematics 65, Longman Sci. (1993).
- [18] R. Livné *Cubic Exponential Sums and Galois Representations*, Contemporary Mathematics **67** (1987), 247–261.
- [19] A. Pizer, *An Algorithm for Computing Modular Forms on $\Gamma_0(N)$* , Journal of Algebra **64** (1980), 340–390.
- [20] C. Schoen, *On the Geometry of a Special Determinantal Hypersurface Associated to the Mumford-Horrock Vector Bundle*, J. Reine Angew. Math. **364** (1986), 85–111.
- [21] J.-P. Serre, *Facteurs locaux des fonctions zêta des variétés algébriques (définitions et conjectures)*, Sémin. Delange-Pisot-Poitou **19** (1969/1970).
- [22] G. Shimura, *The Special Values of the Zeta Functions Associated with Hilbert Modular Forms*, Duke Math. J. **45** (1978), No. 3, 637–679.
- [23] H. Shimizu, *On Zeta Functions of Quaternion Algebras*, Annals of Math. **81** (1965), 166–193.
- [24] J. S. Socrates, *The Quaternionic Bridge Between Elliptic Curves and Hilbert Modular Forms*, Ph.D. Thesis, Caltech (1993).
- [25] R. Taylor, *On Galois Representations Associated to Hilbert Modular Forms*, Invent. Math. **98** (1989), 265–280.
- [26] M. F. Vigneras, *Arithmétique des Algèbres de Quaterniones*, Lecture Notes in Math. 800, Springer-Verlag, Berlin Heidelberg (1980).
- [27] G. van der Geer *Hilbert Modular Surfaces*, Ergebnisse der Mathematik und Ihrer Grenzgebiete 16, Springer-Verlag, Berlin New York (1988).
- [28] B. van Geemen, J. Werner, *Nodal quintics in \mathbf{P}^4* , Lecture Notes in Math. 1399, Springer-Verlag Berlin New York (1989).

- [29] J. Werner, B. van Geemen, *New Examples of Threefolds with $c_1 = 0$* , Math. Z. **203** (1990), No. 2, 211–225.
- [30] J. Waldspurger, *Engendrement par des Séries Theta de Certain Espaces de Formes Modulaires*, Invent. Math. **50** (1979), 135–168.
- [31] R. Weissauer, *Stabile Modulformen und Eiseinteineihen*, Lecture Notes in Math. 1219, Springer-Verlag, Berlin New York (1986).

C. CONSANI: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, ON.
M5S 3G3 CA

E-mail address: `kc@math.toronto.edu`

J. SCHOLTEN: DEPARTMENT OF MATHEMATICS, KATHOLIEKE UNIVERSITEIT NIJMEGEN, POST-
BUS 9010, 6500 GL, NIJMEGEN, THE NETHERLANDS

E-mail address: `scholten@sci.kun.nl`